

Rotation Problems in Forestry

Alternative Derivation of the Faustmann Rotation

First, some basic assumptions:

- Start with bare land.
- Timber price (p), replanting costs (c), and the discount rate (δ) are constant over time.
- Even-aged stands of trees.

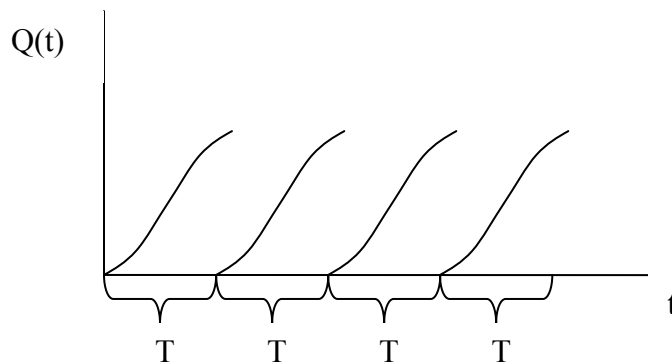
The profits from one harvest are defined as:

$$[pQ(T) - c]e^{-\delta T}$$

With an infinite number of rotations, where T is the time of the rotation (e.g. 40 years), a landowner receives profits every T years, which yields the following profit function:

$$\begin{aligned} & [pQ(T) - c]e^{-\delta T} + [pQ(T) - c]e^{-2\delta T} + [pQ(T) - c]e^{-3\delta T} + \dots \\ \Rightarrow & [pQ(T) - c]e^{-\delta T} \{1 + e^{-\delta T} + e^{-2\delta T} + \dots\} \end{aligned} \quad (a)$$

Graphically, we have the following situation:



If we recognize that the sequence $\{1 + e^{-\delta T} + e^{-2\delta T} + \dots\}$ converges to $(1 - e^{-\delta T})^{-1}$, then we can re-write (a) as:

$$\frac{[pQ(T) - c]e^{-\delta T}}{(1 - e^{-\delta T})} \quad (b)$$

Now, divide the top and bottom of (b) by $e^{-\delta T}$ to get:

$$\frac{[pQ(T) - c]}{(e^{\delta T} - 1)} \quad (c)$$

So, the timber management problem is the following:

$$\max \pi = \frac{[pQ(T) - c]}{(e^{\delta T} - 1)}$$

The first-order condition can be derived by applying the quotient rule:

$$\frac{d\pi}{dT} = \frac{(e^{\delta T} - 1)pQ'(T) - [pQ(T) - c]\delta e^{\delta T}}{(e^{\delta T} - 1)^2} = 0$$

$$\Rightarrow pQ'(T) = \frac{[pQ(T) - c]\delta e^{\delta T}}{e^{\delta T} - 1} \quad (d)$$

Now, divide the top and bottom of the right-hand side by $e^{\delta T}$,

$$\Rightarrow pQ'(T) = \frac{[pQ(T) - c]\delta}{1 - e^{-\delta T}}$$

$$\Rightarrow pQ'(T)(1 - e^{-\delta T}) = [pQ(T) - c]\delta \quad (e)$$

$$\Rightarrow pQ'(T) = [pQ(T) - c]\delta + pQ'(T)e^{-\delta T} \quad (f)$$

Now, let's re-write the last term on the right-hand side of the above equation ($pQ'(T)e^{-\delta T}$). If we multiply equation (d) by $e^{-\delta T}$, we get the following:

$$\Rightarrow pQ'(T)e^{-\delta T} = \frac{[pQ(T) - c]\delta e^{\delta T} e^{-\delta T}}{e^{\delta T} - 1} = \frac{[pQ(T) - c]\delta}{e^{\delta T} - 1} = \delta\pi$$

Plugging this relationship into (f), we get the following:

$$pQ'(T) = [pQ(T) - c]\delta + \delta\pi \quad (g)$$

where the left-hand side of (g) represents the marginal value of waiting, and the right-hand side of (g) represents the marginal cost of waiting. The marginal cost of waiting is comprised of:

$[pQ(T) - c]\delta$: the foregone interest payment if the stand is not cut

$\delta\pi$: cost of delaying future harvests

Note that if the land were re-seeded after cutting, $\delta\pi$ is the foregone rental payment a landowner could charge.

Faustmann rotation vs. Optimal single rotation?

- $T(\text{Faustmann}) < T(\text{single})$
- The marginal cost of waiting under the Faustmann rotation includes an extra cost ($\delta\pi$) to delaying all future harvests.

Derivation of Short-Run Timber Supply Effects

Conrad derives comparative static results in chapter 4.5. I'm going to derive the same results differently than Conrad. In particular, I'm interested in the effects of changes in p , c , and δ on the optimal Faustmann rotation T . I'll make use of the implicit function theorem from calculus, which states that any function $f(x,y)=0$ has a total differential of $0=f_x dx + f_y dy$, so $dy/dx = -f_x/f_y$.

First, we can re-arrange equation (e) in the following way:

$$f(p, T) = pQ'(T)(1 - e^{-\delta T}) - \delta pQ(T) + \delta c = 0$$

The implicit function theorem says the following:

$$\frac{dT}{dp} = -\frac{f_p}{f_T}$$

Conrad has the following assumptions: $Q'(T) > 0$ and $Q''(T) < 0$. So, first calculate f_T :

$$f_T = pQ''(T)(1 - e^{-\delta T}) + \delta pQ'(T)e^{-\delta T} - \delta pQ'(T)$$

$$= \underbrace{pQ''(T)(1 - e^{-\delta T})}_{(-)} + \underbrace{\delta pQ'(T)e^{-\delta T}}_{(+)} - \underbrace{\delta pQ'(T)}_{(-)}$$

Now, calculate f_p :

$$f_p = Q'(T)(1 - e^{-\delta T}) - \delta Q(T)$$

To determine the sign of this equation, recall equation (e) from earlier:

$$\begin{aligned} pQ'(T)(1 - e^{-\delta T}) &= [pQ(T) - c]\delta \\ \Rightarrow Q'(T)(1 - e^{-\delta T}) - \delta Q(T) &= -c\delta / p \end{aligned}$$

Since the parameters c , p , and δ are all positive, then $f_p < 0$. Therefore, using the implicit function theorem, we conclude the following:

$$\frac{dT}{dp} < 0$$

To calculate the sign of dT/dc , we need to calculate f_c :

$$f_c = \delta > 0$$

So, using the implicit function theorem, we conclude the following:

$$\frac{dT}{dc} > 0$$

To calculate the sign of $dT/d\delta$, we need to calculate f_δ :

$$f_\delta = \delta e^{-\delta T} pQ'(T) - pQ(T) + c$$

To determine the sign of this equation, recall equation (d) from earlier:

$$pQ'(T) = \frac{[pQ(T) - c]\delta e^{\delta T}}{e^{\delta T} - 1}$$

Substituting into f_δ , we get the following:

$$f_\delta = \delta e^{-\delta T} \frac{[pQ(T) - c]\delta e^{\delta T}}{e^{\delta T} - 1} - [pQ(T) - c]$$

$$= [pQ(T) - c] \left\{ \frac{\delta^2}{e^{\delta T} - 1} - \frac{e^{\delta T} - 1}{e^{\delta T} - 1} \right\}$$

$$= [pQ(T) - c] \underbrace{\left\{ \frac{\delta^2 + (1 - e^{\delta T})}{e^{\delta T} - 1} \right\}}_{(-)} < 0$$

(-)

So, again appealing to the implicit function theorem, we conclude that

$$\frac{dT}{d\delta} < 0$$

Now, summarize and think about intuition:

- $\frac{dT}{dp} < 0$: If timber price is higher, there is a greater opportunity cost associated with delaying all the harvests, so it's optimal to shorten them.
- $\frac{dT}{dc} > 0$: If the costs of planting are higher, then delaying harvest delays the payment of replanting costs.
- $\frac{dT}{d\delta} < 0$: Higher discount rates imply higher returns on alternative investments to forestry, and so increase the opportunity cost of delaying all future harvests.

Derivation of Long-Run Timber Supply Effects

In the long run, timber supply depends on the number of acres devoted to forestry, which is thought to depend on the forestland rent $\delta\pi$, where:

$$\delta\pi = \frac{[pQ(T) - c]\delta}{e^{\delta T} - 1}$$

Now, some comparative statics reveals the following:

$$\frac{\partial(\delta\pi)}{\partial p} > 0; \quad \frac{\partial(\delta\pi)}{\partial c} < 0; \quad \frac{\partial(\delta\pi)}{\partial \delta} < 0$$

So, what are the long-run supply implications of the following?

- Increase in p :
 - Shortens $T \Rightarrow$ reduces average annual volume $(Q(T)/T) \Rightarrow$ supply reducing
 - Increases forest rent \Rightarrow more land in forest \Rightarrow supply increasing
 - The effect is ambiguous
- Increase in c :
 - Lengthens $T \Rightarrow$ increases average annual volume $(Q(T)/T) \Rightarrow$ supply increasing
 - Decreases forest rent \Rightarrow less land in forest \Rightarrow supply reducing
 - The effect is ambiguous
- Increase in δ :
 - Shortens $T \Rightarrow$ reduces average annual volume $(Q(T)/T) \Rightarrow$ supply reducing
 - Decreases forest rent \Rightarrow less land in forest \Rightarrow supply reducing
 - Supply reducing