Lagrangian Multipliers in Discrete-Time and Derivation of the Fundamental Equation of Renewable Resources

The purpose of this handout is to explicitly go through each step of the derivation of the fundamental equation of renewable resources. This note generally follows the treatment in chapter 1 of Conrad, but I include several steps that Conrad skips.

First, some notation:

X_t : stock of resource in time t

Y_t: level of harvest in time t

$F(X_t)$: biological growth function

$\pi(X_t, Y_t)$: net benefits from resource stock size and harvest in time t.

δ : discount rate

ρ : discount factor

One initial question concerns why X_t is found in $\pi()$? In other words, why are there net benefits from the resource stock? Two common motivations:

- Larger stock is cheaper to harvest (e.g. it's easier to find fish when there are more of them).
- There are non-market values associated with stock size (e.g. birds in a forest stand).

Our problem is to choose the efficient harvest strategy over time, or, the strategy that maximizes the present value of a stream of net benefits over time. The general statement of this problem is as follows:

$$Max \quad \pi = \sum_{t=0}^{T} \rho^t \pi\left(X_t, Y_t\right)$$

$$s.t. \quad X_{t+1} - X_t = F(X_t) - Y_t$$

The solution to this problem will be a harvest level in each period t: $Y_0, Y_1, \ldots, Y_T$. Notice that we have a constraint for each time period t. Therefore, we introduce Lagrange multipliers that are time-specific:

$$L = \sum_{t=0}^{T} \rho^t \left\{ \pi\left(X_t, Y_t\right) + \rho\lambda_{t+1}\left[X_t + F(X_t) - Y_t - X_{t+1}\right]\right\}$$

This is the dynamic Lagrangian for our problem and is equation 1.5 in Conrad. Recall that the Lagrange multiplier has an interpretation as a shadow price on the constraint. Since the constraint equation signifies the dynamics of the resource stock (X_t), each X_t has a Lagrange multiplier λ_t associated with it. Harvest decisions in period t – signified by Y_t – will alter the stock left over in period t+1. The value of an additional unit in t+1 is λ_{t+1}, and this value is discounted one period by ρ to that it is in the same present value terms as $\rho^t \pi(X_t, Y_t)$.

The first order conditions for a maximum are as follows:

$$\frac{\partial L}{\partial Y_t} = \rho^t \left\{ \frac{\partial \pi(\bullet)}{\partial Y_t} - \rho \lambda_{t+1} \right\} = 0 \qquad (1.6)$$

$$\frac{\partial L}{\partial X_t} = \rho^t \left\{ \frac{\partial \pi(\bullet)}{\partial X_t} + \rho \lambda_{t+1} \left[1 + F'(X_t) \right] \right\} - \rho^t \lambda_t = 0 \qquad (1.7)$$

$$\frac{\partial L}{\partial [\rho \lambda_{t+1}]} = \rho^t \left\{ X_t + F(X_t) - Y_t - X_{t+1} \right\} = 0 \qquad (1.8)$$

One tricky part of the first-order conditions to this dynamic optimization problem is the last term in equation (1.7): $\rho^t\lambda_t$. Where does this come from? To see where it comes from, re-write the Lagrangian function as follows:

$$L = \sum_{t=0}^{3} \rho^t \left\{ \pi(X_t, Y_t) + \rho \lambda_{t+1} \left[X_t + F(X_t) - Y_t - X_{t+1} \right] \right\} + \rho^4 \left\{ \pi(X_4, Y_4) + \rho \lambda_5 \left[X_4 + F(X_4) - Y_4 - X_5 \right] \right\}$$

$$+ \rho^5 \left\{ \pi(X_5, Y_5) + \rho \lambda_6 \left[X_5 + F(X_5) - Y_5 - X_6 \right] \right\} + \sum_{t=6}^{T} \rho^t \left\{ \pi(X_t, Y_t) + \rho \lambda_{t+1} \left[X_t + F(X_t) - Y_t - X_{t+1} \right] \right\}$$

All I did was pull out the expressions associated with t=4 and t=5 so that I can see every place that X_5 occurs in the Lagrangian. Now, we can express the derivative of the Lagrangian with respect to X_5 as follows:

$$\frac{\partial L}{\partial X_5} = -\rho^5 \lambda_5 + \rho^5 \left\{ \frac{\partial \pi(\bullet)}{\partial X_5} + \rho \lambda_6 \left[1 + F'(X_5) \right] \right\} = 0$$

Here, you can see that the extra term $\rho^t\lambda_t$ in equation (1.7) comes from the fact that λ_{t+1} is multiplied by both X_t and X_{t+1}.

Now, let's simplify and interpret the three first-order conditions.

(1.9): $\frac{\partial \pi(\bullet)}{\partial Y_t} = \rho \lambda_{t+1}$: marginal value of harvesting an additional unit today must equal the present value of maintaining an additional unit of the resource stock in period t+1.

(1.10): $\lambda_t = \frac{\partial \pi(\bullet)}{\partial X_t} + \rho \lambda_{t+1} \left[1 + F'(X_t) \right]$: marginal value of an additional unit of stock in period t (λ_t) must equal the marginal value of the stock on net benefits in t ($\partial \pi(\bullet)/\partial X_t$) plus the marginal benefit an unharvested unit conveys in period t+1 ($\rho \lambda_{t+1} \left[1 + F'(X_t) \right]$).

(1.11): $X_{t+1} = X_t + F(X_t) - Y_t$: this is the original constraint which says that the resource stock in t+1 equals the stock in t plus biological growth minus harvest.

On our first day of class we introduced the concept of a steady-state equilibrium where all variables reached a constant, sustainable level: $X_{t+1}=X_t=X$; $Y_{t+1}=Y_t=Y$; $\lambda_{t+1}=\lambda_t=\lambda$. The steady-state equilibrium is generally thought to occur when T approaches infinity.

First-order conditions in steady-state:

(1.12): $\frac{\partial \pi(\bullet)}{\partial Y} = \rho\lambda$

Equation (1.13) in the book is derived as follows. Begin with equation (1.10) in steady-state:

$$\lambda = \frac{\partial \pi(\bullet)}{\partial X} + \rho\lambda\left[1+F'(X)\right]$$
$$=> -\frac{\partial \pi(\bullet)}{\partial X} = \rho\lambda\left[1+F'(X)\right] - \lambda$$

and noticing that $\rho = 1/(1+\delta)$ and $-\lambda = -(1+\delta)\rho\lambda$ allows us to re-write this last equation:

(1.13): $-\frac{\partial \pi(\bullet)}{\partial X} = \rho\lambda\left[1+F'(X)-(1+\delta)\right]$

Now, a simple re-write of (1.13) gives us:

(1.15): $-\frac{\partial \pi(\bullet)}{\partial X} = -\rho\lambda\left[\delta - F'(X)\right]$

If we multiply both sides of (1.15) by -1 we get:

$$\frac{\partial \pi(\bullet)}{\partial X} = \rho\lambda\left[\delta - F'(X)\right]$$

Now, substitute (1.12) into the above equation to get the following:

$$\frac{\partial \pi(\bullet)}{\partial X} = \frac{\partial \pi(\bullet)}{\partial Y}\left[\delta - F'(X)\right]$$

A simple rearrangement leads to the <u>fundamental equation of renewable resources (FERR)</u>:

(1.16) $F'(X) + \frac{\partial \pi(\bullet)/\partial X}{\partial \pi(\bullet)/\partial Y} = \delta$

An interpretation of FERR is that the steady-state stock and harvest levels should be set such that the resource's internal rate of return – the left-hand side of (1.16) – equals the discount rate. Note that F'(X) is the marginal net growth rate of the stock while $\frac{\partial \pi(\bullet)/\partial X}{\partial \pi(\bullet)/\partial Y}$ is known as the marginal stock effect. The marginal stock effect is the ratio of the marginal net benefits of an additional unit of stock to the marginal net benefits of an additional unit of harvest. So, FERR encompasses two of the three steady-state first-order conditions (1.12) and (1.13).

The final first-order condition in steady-state ensures that harvest equals biological growth:
(1.14) $Y = F(X)$