AAE / ECON / FOREST 531 (Natural Resource Economics)
Homework #1
Solutions

1. Suppose an individual's utility is expressed as $U = x^{1/2} y^{1/2}$ and their constraint is represented by the following: $2x + 3y = B = 100$.
   a. What level of x and y should the individual consume to maximize utility? What is utility at the optimal level of consumption?

      The Lagrangian can be setup as follows:
      $L = x^{1/2} y^{1/2} + \lambda(100 - 2x - 3y)$
      with the following FOC:
      $$(1): \frac{\partial L}{\partial x} = \frac{1}{2} x^{-1/2} y^{1/2} - 2\lambda = 0 \Rightarrow \lambda = \frac{1}{4}\left(\frac{y}{x}\right)^{1/2}$$
      $$(2): \frac{\partial L}{\partial y} = \frac{1}{2} x^{1/2} y^{-1/2} - 3\lambda = 0 \Rightarrow \lambda = \frac{1}{6}\left(\frac{x}{y}\right)^{1/2}$$
      $$(3): \frac{\partial L}{\partial \lambda} = 100 - 2x - 3y = 0$$
      Noting the relationship between (1) and (2), we can write:
      $$\frac{1}{4}\left(\frac{y}{x}\right)^{1/2} = \frac{1}{6}\left(\frac{x}{y}\right)^{1/2} \Rightarrow 6y = 4x \Rightarrow y = (2/3)x$$
      Substituting into (3):
      $$100 - 2x - 3(2/3)x = 100 - 4x = 0$$
      $$\Rightarrow x^* = 25; y^* = 16.7; U(x^*, y^*) = 20.41$$

   b. What is the shadow price of relaxing the constraint – that is, having B=101 rather than B=100?

      Using the above equations:
      $$\lambda^* = \frac{1}{4}\left(\frac{y^*}{x^*}\right)^{1/2} = \frac{1}{4}\left(\frac{16.7}{25}\right)^{1/2} = 0.2$$

   c. What level of x and y should the individual consume to maximize utility when the constraint is represented by $2x + 3y = B = 101$? What is utility at the optimal level of consumption?

      Using the above equations with B=101:
      $$x^* = 25.25; y^* = 16.8; U(x^*, y^*) = 20.61$$

2. Consider a resource-based economy which can allocate labor (L) to harvest timber (T) or fish (F). Assume the economy faces constant world prices for timber and fish, denoted P_T

and P_F, respectively. Labor is constrained by the following equation: $L = T^2 + F^2/4$. Further, suppose P_T=\$500/ton, P_F=\$1000/ton, and L = 1700 available hours.

a. How should labor be allocated to timber and fish production to maximize the one-period value (V) of resource production? (Note: $V = P_T T + P_F F$).

The relevant problem is the following:

$$Max \quad V = P_T T + P_F$$

$$s.t. \quad L = T^2 + F^2/4$$

Substituting the numerical values, the Lagrangian can be written:

$$L = 500T + 1000F + \lambda(1700 - T^2 - F^2/4)$$

Taking first order conditions:

$$(a)\frac{\partial L}{\partial T} = 500 - 2\lambda T = 0$$

$$(b)\frac{\partial L}{\partial F} = 1000 - (1/2)F\lambda = 0$$

$$(c)\frac{\partial L}{\partial \lambda} = 1700 - T^2 - F^2/4 = 0$$

Solving (a) and (b) for λ, we get:

$$\lambda = 250/T = 2000/F$$

$$=> F = 8T$$

Substituting this expression for F into (c), we get:

$$1700 = T^2 + \frac{64T^2}{4} => T^2 = 100$$

$$=> T = 10 => F = 8(10) = 80$$

b. What is the marginal value (shadow price) of an additional unit of labor?

Since $\lambda = 250/T = 2000/F$, *then λ=250/10=25.*

3. Consider the allocation of a depletable resource over two periods. There are $\bar{Q} = 4$ units of the stock available. The total benefits derived from using the resource are defined as $TB_t = 20q_t - (1/2)q_t^2$, and the total cost of extracting the resource is defined as $TC_t = 5q_t$.

a. What are the values of q_0 and q_1 that maximize the net benefits of using the resource if the discount rate is 10%? What if the discount rate is 5%?

At an optimum, the present value of the marginal net benefits of using the resource should be equal across both periods. In this example,

$$MNB_t = MWTP_t - MC_t = \frac{\partial TB_t}{\partial q_t} - \frac{\partial TC_t}{\partial q_t} = 20 - q_t - 5$$

So, at the optimum with a 10% discount rate:

$$15 - q_0 = (1/1.1)[15 - q_1] => q_0 = 1.36 + 0.91q_1$$

We also know that $q_0 + q_1 = 4$ *at the optimum. So,*

$$1.36 + q_1(1+0.91) = 4 \Rightarrow q_1^* = 1.4; q_0^* = 2.6$$

Doing the same calculations with a 5% discount rate,

$$15 - q_0 = (1/1.05)[15 - q_1] \Rightarrow q_0 = 0.71 + 0.95q_1$$

$$\Rightarrow 0.71 + q_1(1+0.95) = 4 \Rightarrow q_1^* = 1.7; q_0^* = 2.3$$

b. Now suppose there are $\bar{Q} = 30$ units of the stock available. How do the answers to part (a) change? What is your intuition?

If $\bar{Q} = 30$, then the resource is no longer scarce, since MNB_0=0 when q_0=15. So, since it's not optimal to extract more than 15 units in period 0, there will be 15 units left for period 1, and MNB_1=0 when q_1=15. Intuitively, the constraint doesn't matter, so the shadow price of the constraint is zero.

c. Go back to the original assumption of $\bar{Q} = 4$ and a 10% discount rate. What is the price that will be charged in each period? What is the shadow price of the resource in period 0?

$$P_0 = MWTP_0 = 20 - q_0^* = \$17.40$$

$$P_1 = MWTP_0 = 20 - q_1^* = \$18.60$$

From class, we know that the present value of the marginal net benefits from either period equals the shadow price. So, since MNB_0=\$12.40, λ=\$12.40.

d. Again assuming that $\bar{Q} = 4$ with a 10% discount rate, what is the consumer and producer surplus for each period?

Consumer surplus is the area under the MWTP curve and above the price, while producer surplus is the area under the price and above the MC curve. Since we have a simple linear demand curve and constant marginal cost curve:

$$CS_0 = (1/2)(20 - 17.40)(2.6) = \$3.38$$

$$PS_0 = (17.40 - 5)(2.6) = \$32.24$$

$$CS_1 = (1/2)(20 - 18.60)(1.4) = \$0.98$$

$$PS_1 = (18.60 - 5)(1.4) = \$19.04$$

4. Consider the optimal management of a renewable resource where the net benefits of using the resource in time t are defined as $\pi(X_t, Y_t) = 3X_t + 3Y_t$, and the resource has a natural rate of growth described by $F(X_t) = 2X_t[1 - X_t/4]$.

   a. Set up the dynamic Langrangian for this problem and solve for the first-order conditions.

$$L = \sum_{t=0}^{\infty} \rho^t \left\{ 3x_t + 3y_t + \rho\lambda_{t+1} \left[3x_t - 0.5x_t^2 - y_t - x_{t+1} \right] \right\}$$

F.O.C.

$$(i) \frac{\partial L}{\partial Y_t} = \rho^t \{3 - \rho\lambda_{t+1}\} = 0$$

$$(ii) \frac{\partial L}{\partial X_t} = \rho^t \{3 + \rho\lambda_{t+1}[3 - X_t] - \lambda_t\} = 0$$

$$(iii) \frac{\partial L}{\partial \rho\lambda_{t+1}} = \rho^t \left\{3X_t - 0.5X_t^2 - Y_t - X_{t+1}\right\} = 0$$

b. What is the steady state optimum (x*, y*, λ*) if the discount rate (δ) is 0.05? (Hint: use the first-order conditions).

In steady state, we know that the following must hold:

$$X_{t+1} = X_t = X; \quad Y_{t+1} = Y_t = Y; \quad \lambda_{t+1} = \lambda_t = \lambda$$

Using (i) evaluated at steady state:

$$3 - \rho\lambda = 0 \Rightarrow 3 = \lambda/(1+\delta) \Rightarrow \lambda^* = 3(1+\delta) = 3.15$$

Using (ii) evaluated at steady state:

$$3 + \rho\lambda[3 - X]] - \lambda = 0 \Rightarrow X^* = \frac{3 + 3\rho\lambda^* - \lambda^*}{\rho\lambda^*} = \frac{3 + 3(3) - 3.15}{3} = 2.95$$

Using (iii) evaluated at steady state:

$$Y^* = 3X^* - 0.5X^{*2} - X^* = 3(2.95) - 0.5(2.95)^2 - 2.95 = 1.549$$

c. What is the steady state optimum (x*, y*, λ*) if the discount rate (δ) is 0.1?

Using the formulas derived in part (b), $\lambda^* = 3.3; \quad X^* = 2.9; \quad Y^* = 1.695$

d. Interpret the difference between (b) and (c).

A higher discount rate implies that the resource's internal rate of return must increase in order to satisfy the fundamental equation of renewable resources. The numerical example above shows that this is accomplished by lowering X^* and increasing Y^*. Therefore, we can conclude that $F'(X^*)<0$ when δ=0.05, because if $F'(X^*)>0$, then decreases in X^* would only be consistent with decreasing Y^* (think of Figure 1.2 in the book).

e. Using the fundamental equation of renewable resources (eq. 1.16), derive an expression for the resource's internal rate of return as a function of x*.

$$F'(X) + \frac{\partial\pi(.)/\partial X}{\partial\pi(.)/\partial Y} = (2 - X^*) + \frac{3}{3} = 3 - X^*$$

Notice that the marginal stock effect is constant at 1. Therefore, increases in the discount rate imply that the marginal net growth rate (F'(X)) will have to increase in order for the fundamental equation of renewable resources (FERR) to be satisfied. Since F'(X) is always decreasing in X, such an increase implies a lower steady state stock size (X^*) to equate the resource's internal rate of return to the higher discount rate.