1 Basic Attenuation Bias Story

The true model is $c_{it} = \alpha + \beta x^*_{it} + u_{it}$ but the researcher does not observe true income (x^*), rather he observes $x_{it} = x^*_{it} + e_{it}$ where e_{it} has the properties of classical measurement error.

So, when the researcher runs a regression, he estimates:

$$c_{it} = \alpha + \beta x_{it} + u_{it} - \beta e_{it}$$

and so

$$\text{plim}(\hat{\beta}) = \frac{\text{cov}(x, c)}{\text{var}(x)} = \frac{\text{cov}(x^*, c) + \text{cov}(e, c)}{\text{var}(x^*) + \text{var}(e)}$$

and, assuming the measurement error in income is uncorrelated with consumption:

$$= \frac{\text{cov}(x^*, c)}{\text{var}(x^*)} \frac{\text{var}(x^*)}{\text{var}(x^*) + \text{var}(e)} = \beta \frac{\sigma^2_{x^*}}{\sigma^2_{x^*} + \sigma^2_e} = \beta - \beta \frac{\sigma^2_e}{\sigma^2_{x^*} + \sigma^2_e} = \beta(1 - \frac{\sigma^2_e}{\sigma^2_{x^*} + \sigma^2_e}).$$

As the measurement error in income grows, the attenuation bias gets worse.

2 First Difference Attenuation Bias Story

The researcher estimates $c_{it} - c_{it-1} = \beta(x_{it} - x_{it-1}) + (u_{it} - u_{it-1}) - \beta(e_{it} - e_{it-1})$.

Before we start, it is important to note a few definitional facts that build on each other:

1. $\text{cov}(e_{it}, e_{it-1}) = \text{corr}(e_{it}, e_{it-1})\sqrt{\text{var}(e_{it})\text{var}(e_{it-1})} = \sigma^2_e \rho_e$
2. $\text{var}(\Delta x_{it}) = \text{var}(\Delta x^*_{it}) + \text{var}(\Delta e_{it})$
3. $\text{var}(\Delta x^*_{it}) = \text{var}(x^*_{it}) + \text{var}(x^*_{it-1}) - 2\text{cov}(x^*_{it}, x^*_{it-1}) = 2(\sigma^2_{x^*}(1 - \rho_{x^*}))$
4. $\text{var}(\Delta x_{it}) = 2(\sigma^2_{x^*}(1 - \rho_{x^*}) + \sigma^2_e(1 - \rho_e))$

So, you can see that in this case:

$$\text{plim}(\hat{\beta}) = \frac{\text{cov}(\Delta x, \Delta c)}{\text{var}(\Delta x)} = \frac{\text{cov}(\Delta x^*, \Delta c)}{2(\sigma^2_{x^*}(1 - \rho_{x^*}) + \sigma^2_e(1 - \rho_e))}$$

$$= \frac{\text{cov}(\Delta x^*, \Delta c)}{2(\sigma^2_{x^*}(1 - \rho_{x^*}))} \times \frac{2(\sigma^2_{x^*}(1 - \rho_{x^*}))}{2(\sigma^2_{x^*}(1 - \rho_{x^*}) + \sigma^2_e(1 - \rho_e))} = \beta(1 - \frac{\sigma^2_e(1 - \rho_e)}{\sigma^2_{x^*}(1 - \rho_{x^*}) + \sigma^2_e(1 - \rho_e)})$$

and, if we assume that $\rho_e = 0$ then

$$\text{plim}(\hat{\beta}) = \beta(1 - \frac{\sigma^2_e}{\sigma^2_{x^*}(1 - \rho_{x^*}) + \sigma^2_e})$$

So, now, as σ^2_e goes up we still have $\hat{\beta}$ going down, but now we also have the problem that as ρ_{x^*} goes up the attenuation bias gets worse as well. As ρ_{x^*} goes to 1 (which it might) we end up in real trouble.