Constrained Max

$$\underset{\mathbf{x}}{Max}\ f(\mathbf{x}) \text{ s.t. } \mathbf{h}(\mathbf{x}) = 0$$

Set up the Lagrangean eq.
$L = f(\mathbf{x}) + \boldsymbol{\lambda}^T \mathbf{h}(\mathbf{x})$

(1) CQ: $\frac{\partial \mathbf{h}}{\partial \mathbf{x_a}}$ is nonsingular

(2) FONCs ➔ $\mathbf{x}^*$, $\boldsymbol{\lambda}^*$

(3) SONC : symmetric, n.s.d.

$$[\mathbf{s}_b(\mathbf{x}_b^*)^T, \mathbf{I}_{n-m}] \cdot L_{\mathbf{xx}}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \cdot \begin{bmatrix} \mathbf{s}_b(\mathbf{x}_b^*) \\ \mathbf{I}_{n-m} \end{bmatrix}$$

SOSC : above matrix n.d.

Application: cost Min (opposite to Max)

(tech. efficiency ➔ $y = g(\mathbf{x})$)

$$\underset{\mathbf{x}}{Min}\ C = \mathbf{w} \cdot \mathbf{x} \text{ s.t. } y - g(\mathbf{x}) = 0$$

Set up the Lagrangean eq. with one constraint
$L = \mathbf{wx} + \lambda(y - g(\mathbf{x}))$

(1) CQ: $\frac{\partial(y - g(\mathbf{x}))}{\partial x_i} \neq 0$ ➔ $\frac{\partial g(\mathbf{x})}{\partial x_i} = MP_i \neq 0$,
i.e., for at least one input, MP > 0 (because MP ≥ 0)

(2) FONCs ➔ $\mathbf{x}^c(p, \mathbf{w})$, λ^c
(Economic interpretation: isoquant, MRS, shadow value…)

(3) SONC : symmetric, **p**.s.d.

$$[-g_{x_i}^{-1} g_{\mathbf{x}_b}^T, \mathbf{I}_{n-1}] \cdot L_{\mathbf{xx}}(\mathbf{x}^c, \boldsymbol{\lambda}^c) \cdot \begin{bmatrix} -g_{x_i}^{-1} g_{\mathbf{x}_b}^T \\ \mathbf{I}_{n-1} \end{bmatrix}$$

SOSC : above matrix **p**.d.

Note that the SOC is not about the matrix property of $\mathbf{H} = \frac{\partial^2 L}{\partial(\mathbf{x},\boldsymbol{\lambda})^2} = \begin{bmatrix} L_{\mathbf{xx}} & \mathbf{h}_{\mathbf{x}}^T \\ \mathbf{h}_{\mathbf{x}} & \mathbf{0} \end{bmatrix}$, which is called the bordered Hessian, and is always indefinite (due to the **0** submatrix). The reason is that we are not optimizing *L*. For *L*, we only need the FONCs, which give the candidate solutions. The SOCs are related to the original constrained maximization problem.

Math property: sign restrictions of $\frac{\partial^2 L}{\partial(\mathbf{x},\boldsymbol{\lambda})^2} \Leftrightarrow SOC$ in (3)

Conditions that FONCs also sufficient

$f(\mathbf{x})$ concave
$h_j(\mathbf{x})$ quasi-concave
$\boldsymbol{\lambda}^* \geq \mathbf{0}$
-- OR--
$f(\mathbf{x})$ quasi-concave
$f_{\mathbf{x}}(\mathbf{x}^*) \neq \mathbf{0}$
$h_j(\mathbf{x})$ quasi-concave
$\boldsymbol{\lambda}^* \geq \mathbf{0}$

C(**x**) **convex**, satisfied
~ *C(**x**) = **wx**, linear fn is both concave and convex*
$h(\mathbf{x})$ quasi-**convex**, satisfied
~ *h(**x**) = y – g(**x**), (⇔ – g(**x**) quasi-**convex***
*⇔ g(**x**) quasi-**concave**, note that if g(**x**) concave (diminishing MP), ➔ g(**x**) quasi-concave)*
$\lambda^c \geq 0$, satisfied
~ **always** $\lambda^c > 0$, given CQ and **w**>0

Thus: only need quasiconcavity of $g(\mathbf{x})$ for FONCs to be sufficient, other conditions are satisfied by the assumptions in model set-up.

Why do we care about quasi- instead of concave/convex here? A simple answer is that the constrained optimization has less flexibility than the unconstrained optimization case (recall the LeChatelier principle). The conditions required can be more relaxed because the optimization process is over a smaller variable space. Quasi- is weaker than concave/convex. It has the nice property that the level set is convex toward the direction of interest (max or min). Indeed from the graphically interpretation of FONCs, we are finding the optimization point by moving along the isoquant curve (the level set). In order to get an interior solution, all we need is the convexity in level set, thus quasi- conditions are sufficient enough. For computational convenience, it is true that quasi- is less convenient. But in terms of the range of applications, quasi- is much wider and thus is more useful.

If need to do comparative statics, need SOSC (so can do derivatives)…

Math property: *SOSC* in (3)

$$[\mathbf{s}_b(\mathbf{x}_b^c)^T, \mathbf{I}_{n-1}] \cdot L_{\mathbf{xx}}(\mathbf{x}^c, \boldsymbol{\lambda}^c) \cdot \begin{bmatrix} \mathbf{s}_b(\mathbf{x}_b^c) \\ \mathbf{I}_{n-1} \end{bmatrix} \text{ p.d.} \Rightarrow \det\left(\frac{\partial^2 L(\mathbf{x}^c, \lambda^c)}{\partial(\mathbf{x}, \lambda)^2}\right) \neq 0$$

Some major comparative statics results:

$$\begin{bmatrix} \mathbf{x}^c_{(\mathbf{w},y)} \\ \lambda^c_{(\mathbf{w},y)} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{x}^c_{\mathbf{w}} & \mathbf{x}^c_y \\ \lambda^c_{\mathbf{w}} & \lambda^c_y \end{bmatrix} = -\mathbf{H}^{-1} \text{ ; symmetric } \Rightarrow \mathbf{x}^c_y = (\lambda^c_{\mathbf{w}})^T$$

$\mathbf{x}^c_{\mathbf{w}}$: symmetric, n.s.d

Constraint restrictions: $\frac{\partial g}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}^c}{\partial \mathbf{w}} = 0, \quad \frac{\partial g}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}^c}{\partial y} = 1$

Homogeneity restrictions: $\frac{\partial \mathbf{x}^c_{\mathbf{w}}}{\partial \mathbf{w}} \mathbf{w} = 0$

…