Unconstrained Max

$\underset{\mathbf{x}}{Max}\ f(\mathbf{x})$

(1) FONC ➔ $\mathbf{x}^*$

(2) SONC : $f_{\mathbf{xx}}$ n.s.d.

SOSC : $f_{\mathbf{xx}}$ n.d.

Application: Profit Max

(tech. efficiency ➔ $y = g(\mathbf{x})$, plug it in $\pi(\mathbf{x})$)

$\underset{\mathbf{x}}{Max}\ \pi = p \cdot g(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x}$

(1) FONC ➔ $\mathbf{x}^*(p, \mathbf{w})$

(2) SONC : $\pi_{\mathbf{xx}}$ n.s.d.

SOSC : $\pi_{\mathbf{xx}}$ n.d.

Since $\pi_{\mathbf{xx}} = p \cdot g_{\mathbf{xx}}$,
thus $\pi_{\mathbf{xx}}$ n.s.d. $\Leftrightarrow g_{\mathbf{xx}}$ n.s.d.
$\pi_{\mathbf{xx}}$ n.d. $\Leftrightarrow g_{\mathbf{xx}}$ n.d.

Math property:
$f_{\mathbf{xx}}$ n.s.d. $\Leftrightarrow f(\mathbf{x})$ concave
$f_{\mathbf{xx}}$ n.d. $\Rightarrow f(\mathbf{x})$ strictly concave

Conditions that FONCs also sufficient ($\neq$ SOSC)

$f(\mathbf{x})$ concave ($\Leftrightarrow f_{\mathbf{xx}}$ n.s.d., SONC)

$\pi(\mathbf{x})$ concave ($\Leftrightarrow \pi_{\mathbf{xx}}$ n.s.d., SONC, $\Leftrightarrow g_{\mathbf{xx}}$ n.s.d.)
$\Leftrightarrow g(\mathbf{x})$ concave

If need to do comparative statics, need to assume det($\pi_{\mathbf{xx}}$*)* $\neq 0$*, so that we can use the implicit function theorem (*$\pi_{\mathbf{xx}}$ *must be invertible) to do derivatives*:
$\pi_{\mathbf{xx}}$ n.s.d. & *det(*$\pi_{\mathbf{xx}}$*)* $\neq 0 \Leftrightarrow \pi_{\mathbf{xx}}$ n.d., i.e. *SOSC*

$f_{\mathbf{xx}}$ n.d. (➔ $f(\mathbf{x})$ strictly concave)

$\pi_{\mathbf{xx}}$ n.d. (➔ $\pi(\mathbf{x})$ strictly concave)
$\Leftrightarrow g_{\mathbf{xx}}$ n.d. (➔ $g(\mathbf{x})$ strictly concave)

Economic interpretation:
The production function $g(\mathbf{x})$ being concave implies non-increasing MP;
The production function $g(\mathbf{x})$ being strictly concave implies strictly decreasing MP

Some major comparative statics results:

$\mathbf{x}^*_{\mathbf{w}} = [\pi_{\mathbf{xx}}]^{-1}$; symmetric and n.d. ➔ downward sloping input demand, …

$\mathbf{x}^*_p = -[\pi_{\mathbf{xx}}]^{-1}[g_{\mathbf{x}}]^T$; ➔ $g_{\mathbf{x}} \cdot \mathbf{x}^*_p = -g_{\mathbf{x}}[\pi_{\mathbf{xx}}]^{-1}[g_{\mathbf{x}}]^T \geq 0$

$y^*_p = g_{\mathbf{x}} \cdot \mathbf{x}^*_p \geq 0$; ➔ (weakly) upward sloping supply curve, …

$y^*_{\mathbf{w}} = g_{\mathbf{x}}[\pi_{\mathbf{xx}}]^{-1} = -[\mathbf{x}^*_p]^T$; ➔ $y^*_{\mathbf{w}}[g_{\mathbf{x}}]^T = g_{\mathbf{x}}[\pi_{\mathbf{xx}}]^{-1}[g_{\mathbf{x}}]^T \leq 0$

For general model with multiple outputs and inputs: $\mathbf{y}^*_{\mathbf{p}}$ symmetric and p.s.d

….