Lecture 10 CONSUMER THEORY

10.1 REVIEW
[Constrained optimization, primal-dual approach (envelop theorem)...]

10.2 Consumption decisions
Consider a household choosing n consumption goods $\mathbf{x} = (x_1, \ldots, x_n)^T$. The household has preferences represented by a utility function $u(\mathbf{x})$. Also, the household faces the budget constraint: $\mathbf{p}^T \mathbf{x} \le y$ or $\Sigma_i\, p_i\, x_i \le y$ where $y > 0$ is household income, and $\mathbf{p} = (p_1, \ldots, p_n)^T$ is a n-price vector for $\mathbf{x}$, $p_i > 0$ denoting the market price of x_i, $i = 1, \ldots, n$.
Assume that economic rationality for household decisions is represented by the following utility maximization problem

$$V(\mathbf{p}, y) = \underset{\mathbf{x}}{Max} \;\; \{u(\mathbf{x}) : \mathbf{p}^T \mathbf{x} \le y, \mathbf{x} \ge 0, \mathbf{x} \in \mathbf{R}^n\}. \tag{1}$$

This simply states that the households make consumption decisions so as to maximize their utility subject to a budget constraint. Denote the solution of the optimization problem (1) by $\mathbf{x}^*(y, \mathbf{p})$. Here, $\mathbf{x}^*(y, \mathbf{p})$ are utility maximizing decision rules for household consumption. They are also called Marshallian demand functions. The function $V(y, \mathbf{p})$ in (1) is the indirect utility function satisfying $V(y, \mathbf{p}) = u(\mathbf{x}^*(y, \mathbf{p}))$.

Throughout our discussion, we will make the following (intuitive) assumption.
Assumption A1: (non-satiation)

For any consumption bundle $\mathbf{x} \ge 0$ and for every $\varepsilon > 0$, there is a bundle $\mathbf{x}'$ satisfying $u(\mathbf{x}') > u(\mathbf{x})$, $\mathbf{x}' \ge \mathbf{x}$, $\mathbf{x}' \ne \mathbf{x}$, and $\|\mathbf{x} - \mathbf{x}'\| < \varepsilon$.

Non-satiation means that the household can always increase its utility by increasing the consumption of some commodity. This generates the following key result.

Under (A1), the budget constraint is necessary binding at the optimum $\mathbf{x}^*$: $\mathbf{p}^T \mathbf{x}^*(y, \mathbf{p}) = y$.

Proof: Assume that the budget constraint is not binding at $\mathbf{x}^*$: $\mathbf{p}^T \mathbf{x}^*(y, \mathbf{p}) < y$. Under assumption A1 and given $\mathbf{p} > 0$, there exists a consumption bundle $\mathbf{x}'$ such that $\mathbf{p}^T \mathbf{x}' \le y$ and $u(\mathbf{x}') > u(\mathbf{x}^*(y, \mathbf{p}))$. But this contradicts $\mathbf{x}^*(y, \mathbf{p})$ being utility maximizing. Thus, the budget constraint must be binding at $\mathbf{x}^*$.

This means that, under non-satiation (assumption A1), the utility maximization problem (1) can be written as

$$V(\mathbf{p}, y) = \underset{\mathbf{x}}{Max} \;\; \{u(\mathbf{x}) : \mathbf{p}^T \mathbf{x} = y, \mathbf{x} \ge 0, \mathbf{x} \in \mathbf{R}^n\}. \tag{2}$$

Expression (2) is a standard constrained maximization problem. It can be analyzed using the Lagrangean approach. Define the associated Lagrangean

$$L(\mathbf{x}, \lambda, y, \mathbf{p}) = u(\mathbf{x}) + \lambda\, [y - \mathbf{p}^T \mathbf{x}],$$

where λ is a Lagrange multiplier corresponding to the budget constraint $y - \mathbf{p}^T \mathbf{x} = 0$.
We have the following result:

Given $\mathbf{p} > 0$, the constraint qualification (CQ: rank($\mathbf{p}$) = 1) is always satisfied.

Assume that the utility function $u(\mathbf{x})$ is twice continuously differentiable. Then, given an interior solution $\mathbf{x}^*$, the maximization problem (2) implies the FONC

$$\frac{\partial L}{\partial \mathbf{x}} \equiv \frac{\partial u}{\partial \mathbf{x}} - \lambda \mathbf{p} = 0, \tag{3a}$$

$$\frac{\partial L}{\partial \lambda} \equiv y - \mathbf{p}^T \mathbf{x} = 0. \tag{3b}$$

This is a system of $(n+1)$ equations in $(n+1)$ unknowns: $(\mathbf{x}, \lambda)$. Denote the solution of this system of equations by $\mathbf{x}^*(y, \mathbf{p})$ and $\lambda^*(y, \mathbf{p})$. Equation (3a) implies that

$$\frac{\partial u(\mathbf{x}^*(y,\mathbf{p}))}{\partial x_i} = \lambda^*(y, \mathbf{p})\, p_i,\ i = 1, \ldots, n.$$

This gives the following result.

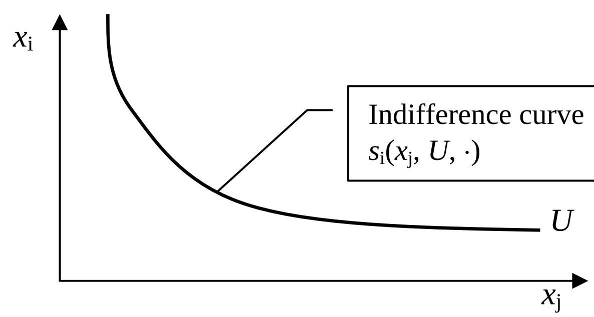
Under non-satiation (A1), the Lagrange multiplier $\lambda^*(y, \mathbf{p})$ is always positive: $\lambda^*(y, \mathbf{p}) > 0$.

Proof: Under non-satiation, $\frac{\partial u(\mathbf{x}^*(y,\mathbf{p}))}{\partial x_i} > 0$, for some i. It follows from (3a) that $\lambda^*(y, \mathbf{p})\, p_i > 0$.

When the price p_i is positive ($p_i > 0$), this implies that $\lambda^*(y, \mathbf{p}) > 0$.

From assumption A1, assume that the utility function is non-satiated in x_i, with $\frac{\partial u}{\partial x_i} > 0$. Define the marginal rate of substitution between x_i and x_j by $MRS_{ij} = \frac{\partial u}{\partial x_j} \Big/ \frac{\partial u}{\partial x_i}$, for $i \neq j$. The MRS_{ij} can be interpreted as the negative of the slope of the indifference curve $x_i = s_i(x_j, U, \cdot)$ obtained from solving the equation $u(x_i, x_j, \cdot) = U$ for x_i, where U is some reference utility level. Indeed, applying the implicit function theorem to $u(x_i, x_j, \cdot) = U$ yields $\frac{\partial s_i}{\partial x_j} = -\frac{\partial u}{\partial x_j} \Big/ \frac{\partial u}{\partial x_i} = -MRS_{ij}$. Figure 10.1 shows the indifference curve $s_i(x_j, U, \cdot)$ giving the set of points $\mathbf{x}$ that keep the household at a given utility level U.

Figure 10.1

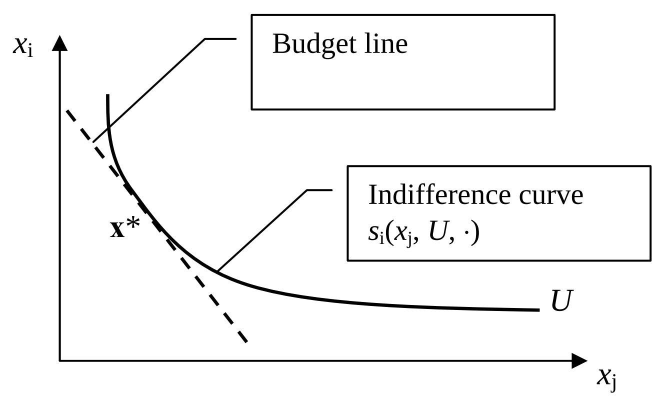


Given $\mathbf{p} > 0$ and $\frac{\partial u}{\partial x_i} > 0$ (under non-satiation with respect to x_i), equation (3a) implies that

$$\frac{\partial u}{\partial x_j} \Big/ \frac{\partial u}{\partial x_i} = \frac{p_j}{p_i},\ \text{for } j \neq i.$$

This states that necessary conditions for an interior solution to the maximization problem (2) are that the marginal rates of substitution, MRS_{ij}, must equal the price ratio, for all $j \neq i$ (figure 10.2).

Figure 10.2



From our earlier discussion of constrained optimization, we know that FONC (3a)-(3b) are sufficient to identify an interior solution **x*** to the maximization problem (2) if

- the utility function is <u>quasi-concave</u> (meaning that the set $\{\mathbf{x}: u(\mathbf{x}) \geq U\}$ is convex),
- $\lambda^* \geq 0$,
- and $\frac{\partial u(\mathbf{x}^*)}{\partial \mathbf{x}} \neq 0$.

But non-satiation (A1) implies that that $\frac{\partial u(\mathbf{x}^*)}{\partial \mathbf{x}} \neq 0$ (since non-satiation requires that $\frac{\partial u}{\partial x_i} > 0$ for some i). Also, we have seen that non-satiation and positive prices imply that the constraint qualification (CQ) holds, and that $\lambda^*(y, \mathbf{p}) > 0$. It follows that, under assumption A1 and positive prices,

- an interior solution **x*** to the utility maximization problem (2) implies FONC (3a)-(3b),
- if the utility function $u(\mathbf{x})$ is <u>quasi-concave</u>, then FONC (3a)-(3b) are <u>sufficient</u> to identify a global interior solution **x*** to the utility maximization problem.

In this context, under <u>quasi-concave</u> utility function, equations (3a)-(3b) are <u>necessary and sufficient</u> to identify an interior solution to the household utility maximization problem (2).

10.2.1 An Example: the Linear Expenditure System (LES)

Consider a household with preferences given by the Stone-Geary utility function

$$u(\mathbf{x}) = \sum_i \beta_i \ln(x_i - \gamma_i),$$

where the β_i satisfy $\beta_i > 0$ and are subject to the normalization rule $\Sigma_i\, \beta_i = 1$. (Note that this utility function is concave in **x**, hence it is quasi-concave in **x**). We assume that $x_i > \gamma_i$, so that the logarithm $\ln(x_i - \gamma_i)$ exists. A utility maximizing household would choose **x** in a way consistent with the maximization problem (2)

$$\underset{\mathbf{x}}{Max} \ \{\sum_i \beta_i \ln(x_i - \gamma_i) : \mathbf{p}^T\mathbf{x} = y, \mathbf{x} \geq 0\}. \tag{2'}$$

The associated Lagrangean is $L = \Sigma_i\, \beta_i \ln(x_i - \gamma_i) + \lambda\, [y - \Sigma_i\, p_i\, x_i]$. From (3), the FONC for an interior solution are

$$\frac{\partial L}{\partial x_i} \equiv \frac{\beta_i}{x_i - \gamma_i} - \lambda p_i = 0,\ i = 1,\ldots, n, \tag{3a'}$$

$$\frac{\partial L}{\partial \lambda} \equiv y - \Sigma_i\, p_i\, x_i = 0. \tag{3b'}$$

Equation (3a') can be written as $\beta_i = \lambda\, p_i\, (x_i - \gamma_i)$. Given the normalization rule $\Sigma_j\, \beta_j = 1$, this implies that $1 = \lambda\, \Sigma_j\, [p_j\, (x_j - \gamma_j)] = \lambda\, [y - \Sigma_j\, (p_j\, \gamma_j)]$. This implies that

$$\lambda^* = \frac{1}{y - \sum_j p_j \gamma_j}.$$

Substituting this result into (3a') gives

$$p_i\, x_i^* = p_i\, \gamma_i + \beta_i\, [y - \Sigma_j\, (p_j\, \gamma_j)],\ i = 1, \ldots, n.$$

This is the <u>linear expenditure system</u>, expressing expenditures on the i-th commodity, $p_i\, x_i^*$ as a <u>linear</u> function of prices p and income y.

<u>Note</u>: the Cobb-Douglas utility function $u(\mathbf{x}) = \Sigma_i\, \beta_i \ln(x_i)$ is a special case of the LES when $\gamma_i = 0$, $i = 1, \ldots, n$.

10.3 Consumption behavior

The analysis just presented means that, under quasi-concave utility $u(\mathbf{x})$, the FONC (3a)-(3b) can identify the utility maximizing demand $\mathbf{x}^*(y, \mathbf{p})$. Here, we want to explore in more detail consumption behavior and the properties of the decision rules $\mathbf{x}^*(y, \mathbf{p})$. For convenience, we focus on the case where the decision rules $\mathbf{x}^*(y, \mathbf{p})$ are differentiable functions.

10.3.1 Comparative statics

Using a "primal approach", we can investigate the properties of the Marshallian demand functions $\mathbf{x}^*(y, \mathbf{p})$ by applying the implicit function theorem to the FONC (3a)-(3b). Given the Lagrangean $L = u(\mathbf{x}) + \lambda\,[y - \mathbf{p}^T\,\mathbf{x}]$, denote the bordered Hessian by

$$\mathbf{H} = \frac{\partial^2 L}{\partial(\mathbf{x}, \lambda)^2} = \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & -\mathbf{p} \\ -\mathbf{p}^T & 0 \end{bmatrix} = \begin{bmatrix} u_{\mathbf{xx}} & -\mathbf{p} \\ -\mathbf{p}^T & 0 \end{bmatrix},$$

evaluated at $(\mathbf{x}^*, \lambda^*)$. The non-singularity of the Jacobian matrix requires that $\det(\mathbf{H}) \neq 0$. This is satisfied under the SOSC. For example, in the case where $n = 2$, the SOSC is: $\det(\mathbf{H}) > 0$. (Note that $\det(\mathbf{H}) \neq 0$ is not implied by the weaker SONC, thus motivating the strengthening of SONC into SOSC). Then, letting $\boldsymbol{\alpha} = (\mathbf{p}, y)$, the implicit function theorem applied to (3a)-(3b) gives

$$\begin{bmatrix} \mathbf{x}^*_{\boldsymbol{\alpha}} \\ \lambda^*_{\boldsymbol{\alpha}} \end{bmatrix} = -\mathbf{H}^{-1} \begin{bmatrix} L_{\mathbf{x}\boldsymbol{\alpha}} \\ L_{\lambda\boldsymbol{\alpha}} \end{bmatrix},$$

or

$$\begin{bmatrix} \mathbf{x}^*_{\mathbf{p}} & \mathbf{x}^*_y \\ \lambda^*_{\mathbf{p}} & \lambda^*_y \end{bmatrix} = -\mathbf{H}^{-1} \begin{bmatrix} -\lambda \mathbf{I}_n & 0 \\ -\mathbf{x}^T & 1 \end{bmatrix}. \tag{4}$$

The comparative static results (4) reflects all the implications of economic rationality for the properties of the (differentiable) Marshallian demand functions $\mathbf{x}^*(y, \mathbf{p})$. However, they take a form that is not very intuitive nor convenient for empirical work. Below, we look for simpler forms that would give some insights on the properties of $\mathbf{x}^*(y, \mathbf{p})$.

10.3.2 Adding-up restrictions

We have seen that under non-satiation (A1), the budget constraint is always binding. This implies that the following relationship always holds

$$\textstyle\sum_j p_j\, x_j^*(y, \mathbf{p}) = y. \tag{5}$$

Differentiating (5) with respect to $(\mathbf{p}, y)$ gives

$$x_i^* + \sum_j p_j \cdot \frac{\partial x_j^*}{\partial p_i} = 0,\ i = 1, \ldots, n, \tag{6a}$$

and

$$\sum_j p_j \cdot \frac{\partial x_j^*}{\partial y} = 1. \tag{6b}$$

Equations (6a) and (6b) are the "adding-up" or "aggregation" restrictions, which must be satisfied if the budget constraint is always binding. Equation (6a) is sometimes called the "Cournot aggregation restriction", while (6b) is called the "Engel aggregation restriction."

Note that the Cournot aggregation restriction (6a) can alternatively be expressed in terms of demand elasticities

$$\frac{p_i x_i^*}{y} + \sum_j \frac{p_j x_j^*}{y} \cdot \frac{\partial \ln(x_j^*)}{\partial \ln(p_i)} = 0, \; i = 1, \ldots, n,$$

where $\frac{p_i x_i^*}{y}$ is the budget share for the *i*-th commodity, and $\frac{\partial \ln(x_j^*)}{\partial \ln(p_i)}$ is the Marshallian price elasticity of x_j* with respect to p_i. Similarly, the Engel aggregation restriction (6b) can be alternatively expressed as

$$\sum_j \frac{p_j x_j^*}{y} \cdot \frac{\partial \ln(x_j^*)}{\partial \ln(y)} = 1,$$

where $\frac{\partial \ln(x_j^*)}{\partial \ln(y)}$ is the income elasticity for the *j*-th commodity. Then, the Engel adding-up restriction states that the weighted sum of the income elasticities across all commodities must be equal to 1, with budget shares as weights.

10.3.3 Homogeneity property

Consider the case where there is proportional change in all prices **p** and income *y*. Then, the budget constraint becomes $(k\,\mathbf{p})^T \mathbf{x} \leq k\, y$ for some $k > 0$, which is equivalent to $\mathbf{p}^T \mathbf{x} \leq y$. This makes it clear that a proportional change in (**p**, *y*) has no effect on the budget constraint. But (**p**, *y*) show up only in the budget constraint in (1) or (2). It follows that a proportional change in (**p**, *y*) would have no effect on consumption behavior, i.e. that

$$\mathbf{x}^*(\mathbf{p}, y) = \mathbf{x}^*(k\, y, k\, \mathbf{p}) \text{ for all } k > 0.$$

This implies that the Marshallian demand functions **x***(*y*, **p**) are homogeneous of degree zero in (*y*, **p**). Intuitively, it means that only changes in relative prices or income are expected to affect Marshallian consumption behavior.

If **x***(*y*, **p**) is continuously differentiable, from Euler equation, this generates the following homogeneity restrictions

$$\frac{\partial x_i^*}{\partial y} \cdot y + \sum_j \frac{\partial x_i^*}{\partial p_j} \cdot p_j = 0, \; i = 1, \ldots, n, \qquad (7)$$

or

$$\frac{\partial \ln(x_i^*)}{\partial \ln(y)} + \sum_j \frac{\partial \ln(x_i^*)}{\partial \ln(p_j)} = 0, \; i = 1, \ldots, n,$$

where $\frac{\partial \ln(x_i^*)}{\partial \ln(y)}$ is the income elasticity for the *i*-th commodity, and $\frac{\partial \ln(x_i^*)}{\partial \ln(p_j)}$ is the price elasticity of the *i*-th Marshallian demand with respect to p_j. Then, there are *n* homogeneity restrictions stating that, for each Marshallian demand function, the sum of the income and (own and cross) price elasticities must be equal to 0.

Note: The homogeneity of degree zero of **x***(*y*, **p**) can be imposed by writing **x***(*y*, **p**) as $\mathbf{x}^*(\frac{\mathbf{p}}{y})$. Indeed, this guarantees that a proportional change in (**p**, *y*) would have no effect on Marshallian demands. This suggests a simple way to impose the homogeneity restrictions on the specification of Marshallian demands: use normalized prices ($\frac{\mathbf{p}}{y}$), where all prices are deflated by consumer income *y*.

Note: Recall the definition of the indirect utility function $V(y, \mathbf{p}) = u(\mathbf{x}^*(y, \mathbf{p}))$. Then, the homogeneity of degree zero of **x***(*y*, **p**) implies that indirect objective function *V*(*y*, **p**) is also homogeneous of degree zero in (*y*, **p**). As just suggested, this homogeneity property can be imposed by specifying the indirect objective function using normalized prices: $V(y, \mathbf{p}) = V(\frac{\mathbf{p}}{y})$.

10.3.4 The primal-dual approach

To gain additional insights into consumption behavior, consider the primal-dual approach discussed earlier. It focuses on the properties of the indirect utility function $V(y, \mathbf{p})$. In the context of the Lagrangean $L(\mathbf{x}, \lambda, y, \mathbf{p}) = u(\mathbf{x}) + \lambda (y - \mathbf{p}^T \mathbf{x})$, the primal-dual approach provides three important results:

1. $V(y, \mathbf{p}) = L(\mathbf{x}^*(y, \mathbf{p}), \lambda^*(y, p), y, p)$.
2. The envelope theorem: $\frac{\partial V}{\partial \boldsymbol{\alpha}} = \frac{\partial L}{\partial \boldsymbol{\alpha}(\mathbf{x}^*, \lambda^*, \boldsymbol{\alpha})}$, where $\boldsymbol{\alpha} = (\mathbf{p}, y)$
3. $V_{\alpha\alpha} - L_{\alpha\alpha} = [L_{\alpha\mathbf{x}}, L_{\alpha\lambda}] \begin{bmatrix} \mathbf{x}_{\boldsymbol{\alpha}}^* \\ \lambda_{\boldsymbol{\alpha}}^* \end{bmatrix}$ = symmetric, positive, semi-definite matrix subject to constraint, where $\boldsymbol{\alpha} = (\mathbf{p}, y)$ and

   $\mathbf{u}^T [V_{\alpha\alpha} - L_{\alpha\alpha}] \mathbf{u} = \mathbf{u}^T [L_{\alpha\mathbf{x}}, L_{\alpha\lambda}] \begin{bmatrix} \mathbf{x}_{\boldsymbol{\alpha}}^* \\ \lambda_{\boldsymbol{\alpha}}^* \end{bmatrix} \mathbf{u} \geq 0$, for all $(n+1)\times 1$ vectors $\mathbf{u}$ satisfying

   $\mathbf{h}_\alpha \mathbf{u} = [-\mathbf{x}^T, 1] \mathbf{u} = 0$.

With $L = u(\mathbf{x}) + \lambda (y - \mathbf{p}^T \mathbf{x})$, result 2 (the envelope theorem) gives

$$\frac{\partial V}{\partial y} = \lambda^*(y, \mathbf{p}),$$

which identifies the Lagrange multiplier λ^* as measuring the marginal utility of income, and

$$\frac{\partial V}{\partial p_i} = -\lambda^*(y, \mathbf{p})\, x_i^*(y, \mathbf{p}),\ i = 1, \ldots, n.$$

Combining these two expressions yields the following important result.

$$x_i^*(y, \mathbf{p}) = -\frac{\partial V}{\partial p_i} \Big/ \frac{\partial V}{\partial y},\ i = 1, \ldots, n. \qquad (8)$$

Expression (8) is called Roy's identity. It conveniently expresses the Marshallian demand functions x_i^* as a ratio of derivatives of the indirect utility function $V(y, \mathbf{p})$.

In addition, given $L = u(\mathbf{x}) + \lambda (y - \mathbf{p}^T \mathbf{x})$ and $\boldsymbol{\alpha} = (\mathbf{p}, y)$, we have $L_{\alpha\alpha} = 0$, $[L_{\alpha\mathbf{x}}, L_{\alpha\lambda}] = \begin{bmatrix} L_{\mathbf{px}} & L_{\mathbf{p}\lambda} \\ L_{y\mathbf{x}} & L_{y\lambda} \end{bmatrix} = \begin{bmatrix} -\lambda^* \mathbf{I}_n & -\mathbf{x}^* \\ 0 & 1 \end{bmatrix}$. Then, result 3 gives

$$V_{\alpha\alpha} = \begin{bmatrix} -\lambda^* \mathbf{I}_n & -\mathbf{x}^* \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{\mathbf{p}}^* & \mathbf{x}_y^* \\ \lambda_{\mathbf{p}}^* & \lambda_y^* \end{bmatrix} = \text{symmetric, positive, semi-definite matrix subject to constraint,} \qquad (9a)$$

and

$$\mathbf{u}^T V_{\alpha\alpha} \mathbf{u} = \mathbf{u}^T \begin{bmatrix} -\lambda^* \mathbf{I}_n & -\mathbf{x}^* \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{\mathbf{p}}^* & \mathbf{x}_y^* \\ \lambda_{\mathbf{p}}^* & \lambda_y^* \end{bmatrix} \mathbf{u} \geq 0, \text{ for all } (n+1)\times 1 \text{ vectors } \mathbf{u} \text{ satisfying}$$

$$[-(\mathbf{x}^*)^T, 1]\, \mathbf{u} = 0. \qquad (9b)$$

Equation (9a) states that the matrix $V_{\alpha\alpha}$ is positive semi-definite subject to constraint. This implies that the indirect objective function $V(\frac{\mathbf{p}}{y})$ is quasi-convex in normalized prices $\frac{\mathbf{p}}{y}$.

In addition, equations (9a)-(9b) can be used to obtain the following important result.

The Slutsky matrix:
The $(n \times n)$ matrix $[\mathbf{x_p}^* + \mathbf{x}_y^* \cdot (\mathbf{x}^*)^T]$ is symmetric, negative semi-definite. (10)

Proof: Premultiplying (9a) by $[\mathbf{I}_n, \mathbf{x}^*]$ and postmultiplying it by the transpose $\begin{bmatrix} \mathbf{I}_n \\ (\mathbf{x}^*)^T \end{bmatrix}$ gives

$\{-\lambda^* [\mathbf{x_p}^* + \mathbf{x}_y^* \cdot (\mathbf{x}^*)^T]\}$. Since the matrix in (9a) is symmetric, the $(n \times n)$ matrix $[\mathbf{x_p}^* + \mathbf{x}_y^* \cdot (\mathbf{x}^*)^T]$ is also symmetric.

To prove the negative semi-definiteness, let $\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$, where $\mathbf{u}_1$ is a $(n \times 1)$ vector, and $\mathbf{u}_2$ is a scalar. The constraint $[-(\mathbf{x}^*)^T, 1]\, \mathbf{u} = 0$ in (9b) implies that $\mathbf{u}_2 = \mathbf{x}^T \mathbf{u}_1$, or

$\mathbf{u} = \begin{bmatrix} \mathbf{I}_n \\ (\mathbf{x}^*)^T \end{bmatrix} \mathbf{u}_1$. It follows from (9b) that

$\mathbf{u}_1^T [\mathbf{I}_n, \mathbf{x}^*] \begin{bmatrix} -\lambda^* \mathbf{I}_n & -\mathbf{x}^* \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_\mathbf{p}^* & \mathbf{x}_y^* \\ \lambda_\mathbf{p}^* & \lambda_y^* \end{bmatrix} \begin{bmatrix} \mathbf{I}_n \\ (\mathbf{x}^*)^T \end{bmatrix} \mathbf{u}_1 \geq 0$, for all $(n \times 1)$ vectors $\mathbf{u}_1$,

or $\mathbf{u}_1^T [-\lambda^* \mathbf{I}_n] [\mathbf{x_p}^* + \mathbf{x}_y^* \cdot (\mathbf{x}^*)^T] \mathbf{u}_1 \geq 0$, for all $(n \times 1)$ vectors u_1,
or, given $\lambda^* > 0$,

$$\mathbf{u}_1^T [\mathbf{x_p}^* + \mathbf{x}_y^* \cdot (\mathbf{x}^*)^T] \mathbf{u}_1 \leq 0, \text{ for all } (n \times 1) \text{ vectors } \mathbf{u}_1,$$

which proves that the matrix $[\mathbf{x_p}^* + \mathbf{x}_y^* \cdot (\mathbf{x}^*)^T]$ is negative semi-definite.

The matrix $[\mathbf{x_p}^* + \mathbf{x}_y^* \cdot (\mathbf{x}^*)^T]$ identified in (10) is called the Slutsky matrix. The symmetry and negative semi-definiteness of the Slutsky matrix are called the integrability conditions of consumer demand. They are the implications of utility maximizing behavior for the properties of the Marshallian demand function $\mathbf{x}^*(y, \mathbf{p})$. Such conditions must be satisfied for consumer decision rules to be consistent with utility maximization. For example, finding empirical evidence against the symmetry or negative semi-definiteness of the Slutsky matrix would be sufficient to conclude that observed household decisions rules are inconsistent with utility maximization.

Equation (10) implies two sets of restrictions on Marshallian demands.

- The symmetry in (10) implies $n(n-1)/2$ symmetry restrictions

$$\frac{\partial x_i^*}{\partial p_j} + \frac{\partial x_i^*}{\partial y} \cdot x_j^* = \frac{\partial x_j^*}{\partial p_i} + \frac{\partial x_j^*}{\partial y} \cdot x_i^*, \text{ for all } i \neq j. \quad (11a)$$

- The negative semi-definiteness in (10) implies the sign restrictions

$$\frac{\partial x_i^*}{\partial p_i} + \frac{\partial x_i^*}{\partial y} \cdot x_i^* \leq 0, \; i = 1, .., n. \quad (11b)$$

Equation (11a) imposes restrictions on cross-price effects. It can alternatively be expressed in terms of elasticities

$$\frac{\partial \ln(x_i^*)}{\partial \ln(p_j)} + \frac{\partial \ln(x_i^*)}{\partial \ln(y)} \cdot \frac{p_j x_j^*}{y} = \frac{\partial \ln(x_j^*)}{\partial \ln(p_i)} + \frac{\partial \ln(x_j^*)}{\partial \ln(y)} \cdot \frac{p_i x_i^*}{y}, \text{ for all } i \neq j,$$

where $\frac{\partial \ln(x_i^*)}{\partial \ln(p_j)}$ is the price elasticity of x_i^* with respect to p_j, $\frac{\partial \ln(x_i^*)}{\partial \ln(y)}$ is the income elasticity of x_i^*, and $\frac{p_i x_i^*}{y}$ is the budget share for the i-th commodity.

Equation (11b) imposes restrictions on own-price effects. It can be expressed alternatively in terms of elasticities

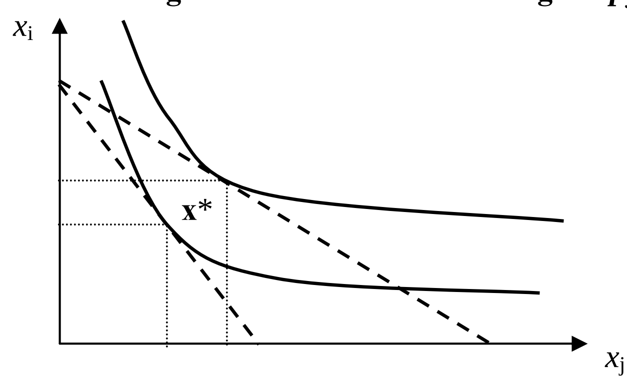
$$\frac{\partial \ln(x_i^*)}{\partial \ln(p_i)} + \frac{\partial \ln(x_i^*)}{\partial \ln(y)} \cdot \frac{p_i x_i^*}{y} \leq 0, \; i = 1, .., n.$$

Equation (11b) implies that $\frac{\partial x_i^*}{\partial p_i} \leq -\frac{\partial x_i^*}{\partial y} \cdot x_i^*$. This means that Marshallian own-price effects $\frac{\partial x_i^*}{\partial p_i}$ are necessarily negative if $\frac{\partial x_i^*}{\partial y} > 0$. This gives the following important result:

Positive income effects ($\frac{\partial x_i^*}{\partial y} > 0$) are sufficient to guarantee that Marshallian own price effects are negative $\frac{\partial x_i^*}{\partial p_i} < 0$, $i = 1, \ldots, n$ (i.e. that Marshallian demands are "downward sloping").

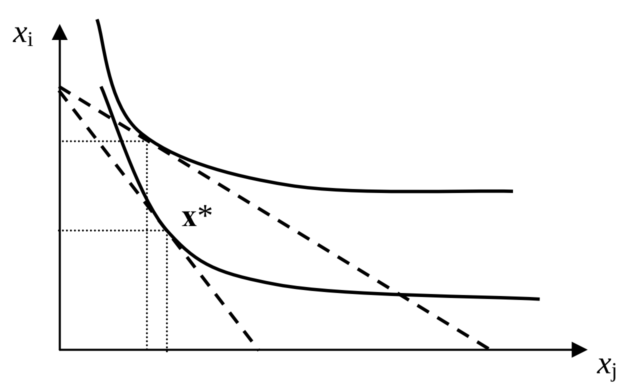
This is illustrated in figure 10.3.

Figure 10.3: Effects of a change in p_j



However, in general, the theory doe not imply that Marshallian own-price effects are always negative. Indeed, there are situations where a higher price can stimulate demand, $\frac{\partial x_i^*}{\partial p_i} > 0$. The commodities exhibiting this property are called Giffen goods. From equation (11b), Giffen goods (satisfying $\frac{\partial x_i^*}{\partial p_i} > 0$) can exist only if income effects $\frac{\partial x_i^*}{\partial y}$ are negative and sufficiently large. This is illustrated in figure 10.4.

Figure 10.4: A Giffen good facing a change in p_j



While the theory indicates that Giffen goods can exist, the empirical evidence suggests that they are rather rare. In other words, while the theory does not imply that all Marshallian demand functions are necessarily "downward sloping," it appears that most Marshallian demands exhibit negative own-price effects…

10.4 Household Production

So far, we have considered households involved only in consumption activities. We now extent this analysis by considering that households can also get involved in production activities. Let $\mathbf{z}$ denote a ($m\times 1$) vector of netputs involved in household production activities, where outputs are positive and inputs are negative. Let $\mathbf{z} = (\mathbf{z}_m, \mathbf{z}_n)$, where $\mathbf{z}_m$ is a vector of market goods with corresponding market price $\mathbf{q}$, and $\mathbf{z}_n$ is a vector of non-market goods (i.e., goods that are not exchanged on a market place). The household production technology is represented by the set $\mathbf{F}$, where $(\mathbf{z}, \mathbf{x}) \in \mathbf{F}$. In the presence of market activities in production, household total income becomes: $y + \mathbf{q}^T \mathbf{z}_m$, where y denotes exogenous income, and $\mathbf{q}^T \mathbf{z}_m$ denotes the profit from marketed production activities. Then, the household budget constraint becomes: $\mathbf{p}^T \mathbf{x} \leq y + \mathbf{q}^T \mathbf{z}_m$. The household utility maximization problem becomes

$$V(y, \mathbf{p}, \mathbf{q}) = \underset{\mathbf{x},\mathbf{z}}{Max} \ \{u(\mathbf{x}, \mathbf{z}_n): \mathbf{p}^T\mathbf{x} \leq y + \mathbf{q}^T\mathbf{z}_m, (\mathbf{z}, \mathbf{x}) \in \mathbf{F}\}, \tag{12a}$$

which has for solution the Marshallian decision rules $\mathbf{x}^+(y, \mathbf{p}, \mathbf{q})$ and $\mathbf{z}^+(y, \mathbf{p}, \mathbf{q})$.

Note that this optimization problem can be decomposed into two stages: first choose $\mathbf{z}_m$; second, choose $(\mathbf{x}, \mathbf{z}_n)$. Then, the household utility maximization (12a) can be equivalently written as

$$V(y, \mathbf{p}, \mathbf{q}) = \underset{\mathbf{x},\mathbf{z}_n}{Max} \ \{\underset{\mathbf{z}_m}{Max} \ \{u(\mathbf{x}, \mathbf{z}_n): \mathbf{p}^T\mathbf{x} \leq y + \mathbf{q}^T\mathbf{z}_m, (\mathbf{z}, \mathbf{x}) \in \mathbf{F}\}\}. \tag{12b}$$

The first stage maximization is

$$\underset{\mathbf{z}_m}{Max} \ \{u(\mathbf{x}, \mathbf{z}_n): \mathbf{p}^T\mathbf{x} \leq y + \mathbf{q}^T\mathbf{z}_m, (\mathbf{z}, \mathbf{x}) \in \mathbf{F}\}. \tag{13a}$$

But under non-satiation (A1), this must necessarily imply the following **profit maximization** problem

$$\pi(\mathbf{q}, \mathbf{z}_n, \mathbf{x}) = \underset{\mathbf{z}_m}{Max} \ \{\mathbf{q}^T\mathbf{z}_m, (\mathbf{z}, \mathbf{x}) \in \mathbf{F}\}, \tag{13b}$$

which has for solution the profit maximizing decision rules $\mathbf{z}_m^*(\mathbf{q}, \mathbf{z}_n, \mathbf{x})$, where $\pi(\mathbf{q}, \mathbf{z}_n, \mathbf{x}) = \mathbf{q}^T \mathbf{z}_m^*(\mathbf{q}, \mathbf{z}_n, \mathbf{x})$ is the indirect profit function. Indeed, if the choice of $\mathbf{z}_m$ did not satisfy (13b), this would necessarily be inconsistent with (12b) since there would exist alternative production decisions that can increase household income, thus making the household better off.

Then, under non-satiation (A1), the second stage maximization in (12b) becomes

$$\underset{\mathbf{x},\mathbf{z}_n}{Max} \ \{u(\mathbf{x}, \mathbf{z}_n): \mathbf{p}^T\mathbf{x} \leq y + \pi(\mathbf{q}, \mathbf{z}_n, \mathbf{x}), \mathbf{x} \geq 0\}, \tag{13c}$$

which has for solution $\mathbf{x}^+(y, \mathbf{p}, \mathbf{q})$ and $\mathbf{z}_n^+(y, \mathbf{p}, \mathbf{q})$. Note that $(\mathbf{z}_n, \mathbf{x})$ appear in the budget constraint in (13c) through the indirect profit function $\pi(\mathbf{q}, \mathbf{z}_n, \mathbf{x})$. This implies that the consumer goods $\mathbf{x}$ have an explicit market price $\mathbf{p}$ as well as an implicit or shadow price $\frac{\partial \pi}{\partial \mathbf{x}}$. Similarly, the non-market goods $\mathbf{z}_n$ have a shadow price $\frac{\partial \pi}{\partial \mathbf{z}_n}$. This illustrates that the household production model is relevant in analyzing the allocation of non-market goods.

This has important implications.

1. Under non-satiation (A1), household utility maximization implies **profit maximization** with respect to the market production activities. This indicates that profit maximization is a relevant economic concept for households, even in the presence of non-market goods.

This is an intuitive result: as long as the marginal utility of income is positive, the household have an incentive to maximize income, and thus to maximize profit (assuming that it is the "residual claimant").

2. The solution to the first stage profit-maximization problem $\mathbf{z}_m^*(\mathbf{q}, \mathbf{z}_n, \mathbf{x})$ is independent of consumer prices **p** and of consumer income *y*. This indicates that, in a household context, production decisions for marketed production activities are **separable** from the consumption decisions.
3. The equivalence between (12a) and (13a)-(13c) implies that
$$\mathbf{z}_m^+(y, \mathbf{p}, \mathbf{q}) = \mathbf{z}_m^*(\mathbf{q}, \mathbf{z}_n^+(y, \mathbf{p}, \mathbf{q}), \mathbf{x}^+(y, \mathbf{p}, \mathbf{q})). \quad (14)$$
4. Non-market goods have a shadow price represented by $\frac{\partial \pi}{\partial z_n}$ measuring their opportunity cost and the extent of their resource scarcity.

10.4.1 The Case of Time and Labor Allocation

As an illustration, we now consider a slight modification of the household production model to analyze time and labor allocation in the household. We consider the case of a household allocating its time between three possible activities:

- leisure (s),
- work within the household ($\mathbf{z}_n$),
- work as wage labor outside the household ($\mathbf{z}_w$).

Let s denote leisure time. It generates household utility. Thus, the household utility function is $u(\mathbf{x}, s)$, where **x** denotes other consumption goods.

Work within the household is a non-market activity and is denoted by z_n. In general, z_n is an input in the household production process. Assume that the outputs of this process are market goods $\mathbf{z}_h$ with corresponding market prices $\mathbf{q}_h$. The household production technology is represented by the set **H**, with $(\mathbf{z}_h, z_n) \in \mathbf{H}$.

Another market activity is selling a quantity z_w of household labor on the labor market at a wage rate q_w.

Thus, the marketed production activities are $\mathbf{z}_m = (z_w, \mathbf{z}_h)$, with corresponding market prices $\mathbf{q} = (q_w, \mathbf{q}_h)$. The household feasible set for $(\mathbf{z}, s)$ is
$$\mathbf{F} = \{(z_w, \mathbf{z}_h, z_n, s): z_w + z_n + s = T, (\mathbf{z}_h, z_n) \in \mathbf{H}, \mathbf{z} \geq 0\},$$
where T denotes the total household time available, and $z_w + z_n + s = T$ is the household time constraint.

The household budget constraint is
$$\mathbf{p}^T \mathbf{x} \leq y + \mathbf{q}^T \mathbf{z}_m = y + q_w z_w + \mathbf{q}_h \mathbf{z}_h.$$

The utility maximizing household solves the optimization problem
$$V(y, \mathbf{p}, \mathbf{q}) = \underset{\mathbf{x},\mathbf{z},s}{Max} \{u(\mathbf{x}, s): \mathbf{p}^T\mathbf{x} \leq y + q_w z_w + \mathbf{q}_h \mathbf{z}_h, (\mathbf{z}, s) \in \mathbf{F}, (\mathbf{x}, s) \geq 0\},$$
or,
$$V(y, \mathbf{p}, \mathbf{q}) = \underset{\mathbf{x},\mathbf{z},s}{Max} \{u(\mathbf{x}, s): \mathbf{p}^T\mathbf{x} \leq y + q_w z_w + \mathbf{q}_h \mathbf{z}_h, z_w + z_n + s = T, (\mathbf{z}_h, z_n) \in \mathbf{H}, (\mathbf{z}, \mathbf{x}, s) \geq 0\},$$
or, after substituting the time constraint assuming that $z_w > 0$,
$$V(y, \mathbf{p}, \mathbf{q}) = \underset{\mathbf{x},\mathbf{z},s}{Max} \{u(\mathbf{x}, s): \mathbf{p}^T\mathbf{x} \leq y + q_w (T - z_n - s) + \mathbf{q}_h \mathbf{z}_h, (\mathbf{z}_h, z_n) \in \mathbf{H}, (\mathbf{z}, \mathbf{x}, s) \geq 0\}, \quad (15a)$$
which has for solution the Marshallian choice functions $\mathbf{x}^+(y, \mathbf{p}, \mathbf{q})$, $s^+(y, \mathbf{p}, \mathbf{q})$ and $\mathbf{z}^+(y, \mathbf{p}, \mathbf{q})$.

Consider a two-stage decomposition of (15a): first choose ($\mathbf{z}_h$, z_n); second choose ($\mathbf{x}$, s)

$$V(y, \mathbf{p}, \mathbf{q}) = \underset{\mathbf{x},s}{Max}\ \{\underset{\mathbf{z}_h,z_n}{Max}\ \{u(\mathbf{x},s) : \mathbf{p}^T\mathbf{x} \leq y + q_w(T - z_n - s) + \mathbf{q}_h\mathbf{z}_h, (\mathbf{z}_h, z_n) \in \mathbf{H}, (\mathbf{z},\mathbf{x},s) \geq 0\}\}, \quad (15b)$$

Under non-satiation, the first-stage optimization in (15b) implies the **profit maximization problem**

$$\pi(\mathbf{q}) = \underset{\mathbf{z}_h,z_n}{Max}\ \{\mathbf{q}_h\mathbf{z}_h - q_w z_n, (\mathbf{z}_h, z_n) \in \mathbf{H}, \mathbf{z} \geq 0\} \quad (15c)$$

which has for solution the profit maximizing decision rules $\mathbf{z}_h^*(\mathbf{q})$ and $z_n^*(\mathbf{q})$, where $\pi(\mathbf{q})$ is the indirect profit function. Note the household labor input z_n is valued at its opportunity cost q_w in (15c).

Under non-satiation (A1), the second stage optimization in (15b) becomes

$$V(y, \mathbf{p}, \mathbf{q}) = \underset{\mathbf{x},s}{Max}\ \{u(\mathbf{x},s) : \mathbf{p}^T\mathbf{x} + q_w s = y + q_w T + \pi(\mathbf{q}), (\mathbf{x},s) \geq 0\}, \quad (15d)$$

which has for solution $\mathbf{x}^+(y, \mathbf{p}, \mathbf{q})$ and $s^+(y, \mathbf{p}, \mathbf{q})$ (since (15a) implies (15d)).

Note that (15d) involves the term: $y + q_w T + \pi(\mathbf{q})$. This term is called the "full income" of the household: $Y = y + q_w T + \pi(\mathbf{q})$. It is the sum of three components: exogenous income y; the opportunity cost of total time, $q_w T$; and profit $\pi(\mathbf{q})$. Full income is the amount of money that is to be allocated to consumer goods $\mathbf{x}$ as well as leisure s (evaluated at its opportunity cost q_w). This suggests that the solution of (15d) can also be expressed as $\mathbf{x}^\#(Y, \mathbf{p}, q_w)$ and $s^\#(Y, \mathbf{p}, q_w)$, where $Y = (y + q_w T + \pi(\mathbf{q}))$ denotes "full income."

Since (15a) implies (15c), the following results must necessarily hold.

$$\mathbf{z}_h^+(y, \mathbf{p}, \mathbf{q}) = \mathbf{z}_h^*(\mathbf{q}),$$

and

$$z_n^+(y, \mathbf{p}, \mathbf{q}) = z_n^*(q).$$

Again, this shows the separability of production decisions from the consumption decisions, as $\mathbf{z}_h^+$ and z_n^+ are independent of the consumption parameters ($\mathbf{p}$, y). Also, it illustrates once more that profit maximization motives can apply to household allocation decisions.

Similarly, since (15a) implies (15d), we obtain the following results.

$$\mathbf{x}^+(y, \mathbf{p}, \mathbf{q}) = \mathbf{x}^\#(y + q_w T + \pi(\mathbf{q}), \mathbf{p}, q_w), \quad (16a)$$

and

$$s^+(y, \mathbf{p}, \mathbf{q}) = s^\#(y + q_w T + \pi(\mathbf{q}), \mathbf{p}, q_w). \quad (16b)$$

Equations (16a)-(16b) show that the profit from production activities $\pi(\mathbf{q})$ behaves like an income effect on consumption/leisure decisions. Such profit/income effects provide significant insights into the analysis of household behavior and time allocation…

Similarly, from (16a)-(16b), the wage rate q_w is found to have **three** effects on consumption/labor allocation: one direct effect and two indirect income effects: one through the time endowment $q_w T$, and one through the profit function $\pi(\mathbf{q})$. Again, such profit/income effects provide significant insights into the effects of the wage rate on household behavior and labor supply. To illustrate, assuming interior solutions, differentiate (16b) with respect to the wage rate q_w. This yields

$$\frac{\partial s^+}{\partial q_w} = \frac{\partial s^\#}{\partial q_w} + \frac{\partial s^\#}{\partial y}\cdot T + \frac{\partial s^\#}{\partial y}\cdot\frac{\partial \pi}{\partial q_w}$$

$$= \frac{\partial s^\#}{\partial q_w} + \frac{\partial s^\#}{\partial y}\cdot T - \frac{\partial s^\#}{\partial y}\cdot z_n^*\,, \tag{17}$$

since $\frac{\partial \pi}{\partial q_w} = -z_n^*$ applying Hotelling's lemma in (15c).

In the case where leisure exhibits positive income effects ($\frac{\partial s^\#}{\partial y} > 0$), the first term in (17), $\frac{\partial s^\#}{\partial q_w}$ (the direct effect), is expected to be negative; the second term, $\frac{\partial s^\#}{\partial y}\cdot T$ (the time endowment effect), is expected to be positive, and the third term, $-\frac{\partial s^\#}{\partial y}\cdot z_n^*$ (the profit effect), is expected to be negative. Thus, the net effect, $\frac{\partial s^+}{\partial q_w}$, can be either positive or negative. For example, it can be negative if the absolute value of the first term, $|\frac{\partial s^\#}{\partial q_w}|$, is relatively large. Alternatively, it can be positive if the second term, $\frac{\partial s^\#}{\partial y}\cdot T$, dominates the other two. In this case, through its time endowment effect, an increase in the wage rate q_w would stimulate the demand for leisure. Intuitively, this would happen when the income elasticity of leisure is "high": a higher wage would increase household income, which would in turn have a large positive effect on the demand for leisure. From the household time constraint, this would imply that the household is spending more time on leisure, and thus less time on work (either on household work or wage labor). This identifies a situation where the wage rate q_w has a negative effect on the supply of household labor. In other words, it is possible for the labor supply function to be "downward sloping"…