Lecture 9 COST MINIMIZATION II

9.1 REVIEW

[Cost minimization – primal approach, duality….]

9.2 Introduction

Consider a firm facing a production technology represented by the production function $y = g(\mathbf{x})$, where y denotes output and $\mathbf{x} = (x_1, \ldots, x_n)$ is a n-vector of inputs. The firm faces market prices $\mathbf{w}$ for its inputs $\mathbf{x}$. Assume that the firm choose inputs so as to minimize cost. This corresponds to the standard cost minimization problem

$$C(y, \mathbf{w}) = \underset{\mathbf{x}}{Min}\{\mathbf{w} \cdot \mathbf{x} : y = g(\mathbf{x}), \mathbf{x} \geq 0, \mathbf{x} \in \mathbf{R}^n\}, \tag{1}$$

which has for solution the cost minimizing input demand functions $\mathbf{x}^c(y, \mathbf{w})$.

Consider the associated Lagrangean $L = \mathbf{w}^T \mathbf{x} + \lambda\, [y - g(\mathbf{x})]$, where λ is a Lagrange multiplier associated with the constraint $y - g(\mathbf{x}) = 0$. Assuming that the constraint qualification CQ holds (i.e. that $\frac{\partial g}{\partial x_1(\mathbf{x}^c)} \neq 0$), the first order necessary condition FONC is

$$w_i = \lambda^c \cdot \frac{\partial g}{\partial x_i(\mathbf{x}^c)},\ i = 1, \ldots, n, \tag{2a}$$

which implies that, at $\mathbf{x}^c(y, \mathbf{w})$,

$$\frac{w_i}{w_j} = \frac{\frac{\partial g}{\partial x_i}}{\frac{\partial g}{\partial x_j}},\ \text{for all } i \neq j. \tag{2b}$$

Also, applying the envelope theorem to $C(y, \mathbf{w})$, we obtain

$$\frac{\partial C}{\partial w_i} = x_i^c(y, \mathbf{w}),\ (\textit{Shephard's lemma}), \tag{3a}$$

$i = 1, \ldots, n$, and

$$\frac{\partial C}{\partial y} = \lambda^c. \tag{3b}$$

Equations (3a)-(3b) appear *quite useful*. First, *Shephard's lemma* in (3a) gives the cost minimizing input demand functions, obtained simply by taking the derivative of the indirect cost function $C(y, \mathbf{w})$ with respect to $\mathbf{w}$. Second, (3b) shows that the Lagrange multiplier λ^c is actually measuring the *marginal cost of production*, $\frac{\partial C}{\partial y(y, \mathbf{w})}$. This indicates that the Lagrange multiplier can provide useful information in economic analysis.

Here, we want to use these results to investigate the linkages between the underlying technology and the properties of $\mathbf{x}^c(y, \mathbf{w})$ and $C(y, \mathbf{w})$, and gain additional insights into production behavior.

9.3 Relationship with profit maximization

For a competitive firm

Consider a competitive firm facing market price p for output y and $\mathbf{w}$ for inputs $\mathbf{x}$. Then, profit maximization for a competitive firm can be written as

$$\underset{y, \mathbf{x}}{Max}\ \{p \cdot y - \mathbf{w} \cdot \mathbf{x} : y = g(\mathbf{x}), \mathbf{x} \geq 0, \mathbf{x} \in \mathbf{R}^n\}, \tag{4a}$$

which has for solution the profit maximizing input demand $\mathbf{x}^*(p, \mathbf{w})$ and output supply $y^*(p, \mathbf{w})$.

Decompose this problem into two stages: in a first stage, choose input $\mathbf{x}$ conditional on y; in a second stage choose y. This can be written as

$$\begin{aligned}
&\underset{y}{Max}\ \underset{\mathbf{x}}{Max}\ \{p\cdot y-\mathbf{w}\cdot\mathbf{x}: y=g(\mathbf{x}), \mathbf{x}\geq 0, \mathbf{x}\in\mathbf{R}^n\}\\
&=\underset{y}{Max}\ \{p\cdot y+\underset{\mathbf{x}}{Max}\ \{-\mathbf{w}\cdot\mathbf{x}: y=g(\mathbf{x}), \mathbf{x}\geq 0, \mathbf{x}\in\mathbf{R}^n\}\}\\
&=\underset{y}{Max}\ \{p\cdot y-\underset{\mathbf{x}}{Min}\ \{\mathbf{w}\cdot\mathbf{x}: y=g(\mathbf{x}), \mathbf{x}\geq 0, \mathbf{x}\in\mathbf{R}^n\}\}\\
&=\underset{y}{Max}\ \{p\cdot y-C(y,\mathbf{w})\}. \qquad (4b)
\end{aligned}$$

It follows that *profit maximization for a competitive firm always implies cost minimization.*

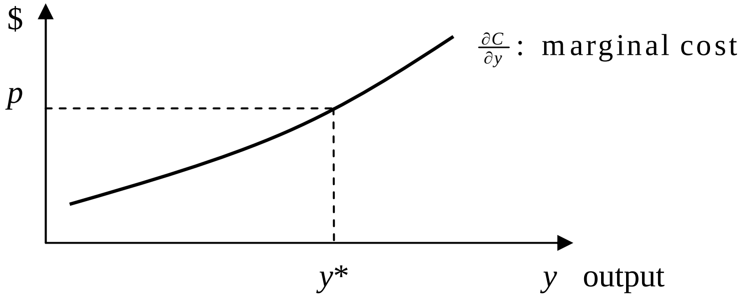
This has two important implications.

- Expression (4b) is a standard unconstrained optimization problem. For an interior solution, $y^* > 0$, the first order necessary condition (FONC) in (4b) is

  $$p=\frac{\partial C}{\partial y(y^*,\mathbf{w})}. \qquad (5)$$

  This means that, in the presence of interior solutions, profit maximization implies *marginal cost pricing: output price must be equal to the marginal cost of production.* In that sense, the marginal cost curve is the supply function for the firm. See figure 9.1.
  In addition, from (4b), the second order necessary condition (SONC) for profit maximization is $\frac{\partial^2 C}{\partial y^2(y^*)} \geq 0$, while the second order sufficient condition (SOSC) is $\frac{\partial^2 C}{\partial y^2(y^*)} > 0$. This means that under profit maximization, *the marginal cost curve cannot be downward sloping with respect to output y*. And if the marginal cost curve is upward sloping with respect to *y*, $\frac{\partial^2 C}{\partial y^2(y^*)} > 0$, then the FONC in (5) is sufficient to identify a local interior solution to the profit maximization problem.
  Finally, *if the production technology exhibits decreasing marginal productivity*, then the production function $g(\mathbf{x})$ is concave, *the cost function C(y,* **w***) is convex in y* (corresponding to a non-decreasing marginal cost curve with respect to *y*: $\frac{\partial^2 C}{\partial y^2} \geq 0$ for all $y > 0$), *and the FONC (5) gives a necessary and sufficient condition for an interior solution y* to the profit maximization problem.*

Figure 9.1



- From equation (4b), the following identities must hold:

  $$\mathbf{x}^*(p,\mathbf{w}) = \mathbf{x}^c(y^*(p,\mathbf{w}),\mathbf{w}). \qquad (6)$$

  Equation (6) states that cost-minimizing behavior is always consistent with profit maximizing behavior for a competitive firm.

Note that $\mathbf{x}^*$ allow adjustments in both $\mathbf{x}$ and y, while $\mathbf{x}^c$ allows adjustments only in inputs $\mathbf{x}$, holding output y constant. As an application of the *LeChatelier principle*, it follows that

$$\left|\frac{\partial \mathbf{x}^*}{\partial \mathbf{w}}\right| - \left|\frac{\partial \mathbf{x}^c}{\partial \mathbf{w}}\right| = \text{a symmetric, positive semi-definite matrix.}$$

This means that $\left|\frac{\partial x_i^*}{\partial w_i}\right| \geq \left|\frac{\partial x_i^c}{\partial w_i}\right| \geq 0$: the magnitude of a quantity response to an input price change tends to be greater under profit maximization compared to cost minimization.

For a monopoly

Consider a monopoly facing market prices $\mathbf{w}$ for inputs $\mathbf{x}$, and a demand function $p(y)$ for its output y. Assume that the demand function $p(y)$ is downward sloping. Then, firm revenue is $R(y) = p(y) \cdot y$, and profit is: $R(y) - \mathbf{w} \cdot \mathbf{x} = p(y) \cdot y - \mathbf{w} \cdot \mathbf{x}$. And profit maximization for a monopoly can be written as

$$\underset{y,\mathbf{x}}{Max} \ \{p(y)\cdot y - \mathbf{w}\cdot\mathbf{x} : y = g(\mathbf{x}), \mathbf{x} \geq 0, \mathbf{x} \in \mathbf{R}^n\}, \tag{7a}$$

which has for solution the monopoly input demand $\mathbf{x}^m(\mathbf{w})$ and output supply $y^m(\mathbf{w})$.

Again, decompose this problem into two stages: in a first stage, choose input $\mathbf{x}$ conditional on y; in a second stage choose y. This can be written as

$$\begin{aligned}
&\underset{y}{Max}\ \underset{\mathbf{x}}{Max}\ \{p(y)\cdot y - \mathbf{w}\cdot\mathbf{x} : y = g(\mathbf{x}), \mathbf{x} \geq 0, \mathbf{x} \in \mathbf{R}^n\}\\
&= \underset{y}{Max}\ \{p(y)\cdot y + \underset{\mathbf{x}}{Max}\ \{-\mathbf{w}\cdot\mathbf{x} : y = g(\mathbf{x}), \mathbf{x} \geq 0, \mathbf{x} \in \mathbf{R}^n\}\}\\
&= \underset{y}{Max}\ \{p(y)\cdot y - \underset{\mathbf{x}}{Min}\ \{\mathbf{w}\cdot\mathbf{x} : y = g(\mathbf{x}), \mathbf{x} \geq 0, \mathbf{x} \in \mathbf{R}^n\}\}\\
&= \underset{y}{Max}\ \{p(y)\cdot y - C(y,\mathbf{w})\}.
\end{aligned} \tag{7b}$$

It follows that *profit maximization for a monopoly always implies cost minimization*.

This has two important implications.

- Expression (7b) is a standard unconstrained optimization problem. For an interior solution, $y^m > 0$, the first order necessary condition (FONC) in (7b) is

  $$p(y^m) + \frac{\partial p}{\partial y(y^m)} \cdot y^m = \frac{\partial C}{\partial y(y^m, \mathbf{w})}. \tag{8}$$

  The left hand side of (8) is marginal revenue with respect to output y: $\frac{\partial R}{\partial y} = p + \frac{\partial p}{\partial y} \cdot y$. Thus, (8) states that, for an interior solution, the profit maximizing monopoly chooses y^m such that *marginal revenue equals marginal cost*.

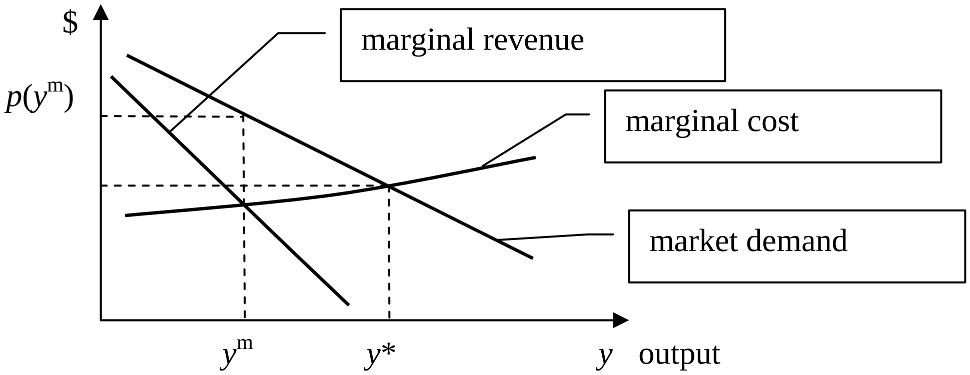
  Note that, with $\frac{\partial p}{\partial y(y^m)} < 0$, equation (8) implies that $p(y^m) > \frac{\partial C}{\partial y(y^m, \mathbf{w})}$. This means that, in the presence of interior solutions, *a monopoly always departs from marginal cost pricing: at y^m, output price exceeds the marginal cost of production*. See figure 9.2. This indicates that a profit maximizing monopoly would sell its output at a higher price compared to a competitive firm. This implies that monopolies would have adverse effects on consumer welfare.
- From equation (7b), the following identities must hold:

  $$\mathbf{x}^m(\mathbf{w}) = \mathbf{x}^c(y^m(\mathbf{w}), \mathbf{w}). \tag{9}$$

  Equation (9) states that cost-minimizing behavior is always consistent with profit maximizing behavior for a monopoly.

Thus, cost minimization can be consistent with firm behavior under alternative market structures (i.e., competitive markets and monopoly). This indicates that cost minimization can apply under more general conditions than profit maximization. As such, it is a weaker concept than profit maximization, with a greater range of applications than profit maximization (e.g., decision making under risk)…

Figure 9.2



9.4 Returns to scale

9.4.1 From the production function

Definition: A production technology exhibits

- *increasing returns to scale* (IRTS) if $g(t \cdot \mathbf{x}) > t \cdot g(\mathbf{x})$ for all $\mathbf{x} \geq 0$ and all $t > 1$,
- *constant returns to scale* (CRTS) if $g(t \cdot \mathbf{x}) = t \cdot g(\mathbf{x})$ for all $\mathbf{x} \geq 0$ and all $t > 0$,
- *decreasing return to scale* (DRTS) if $g(t \cdot \mathbf{x}) < t \cdot g(\mathbf{x})$ for all $\mathbf{x} \geq 0$ and all $t > 1$.

A production technology exhibits *variable return to scale* (VRTS) if it exhibits IRTS and DRTS in different regions **x**.

Definition: A production technology is homogeneous of degree r if the production function is *homogeneous of degree r*, i.e. if it satisfies

$$g(t \cdot \mathbf{x}) = t^r \cdot g(\mathbf{x}), \tag{10}$$

for all $\mathbf{x} \geq 0$ and all $t > 0$.

This has two important implications:

- The production technology exhibits CRTS *if and only if* the production technology is *homogeneous of degree one* (corresponding to $g(\mathbf{x})$ being *homogeneous of degree one in* **x**).
- The production technology exhibits IRTS, CRTS, or DRTS, *if* the production function $g(\mathbf{x})$ is *homogeneous of degree r in* **x**, with $r > 1$, $r = 1$, or $r < 1$, respectively.
  (This follows from (10) since, for any $t > 1$, we have $t^r > t$, $= t$, or $< t$ when, respectively, $r > 1$, $= 1$, or < 1).

Differentiating both sides of (10) with respect to t, evaluated at $t = 1$ gives $\sum_i \frac{\partial g}{\partial x_i} \cdot x_i = r \cdot g(\mathbf{x})$.

This suggests that, for any production function $g(\mathbf{x})$, a ***local*** *measure of the degree of homogeneity of g(***x***)* at point **x** is given by the *scale elasticity* SE

$$SE(\mathbf{x}) = \sum_i \frac{\partial g(\mathbf{x})}{\partial x_i} \cdot \frac{x_i}{g(\mathbf{x})} = \sum_i \frac{\partial \ln(g(\mathbf{x}))}{\partial \ln(x_i)}, \qquad (11)$$

which is the sum of the production elasticities across all inputs. The scale elasticity SE measures the proportional change in output due a (small) proportional change in all inputs.

This has two implications:

- If the production technology is *homogeneous of degree r*, then the scale elasticity $SE(\mathbf{x})$ is equal to *r globally*: $SE(\mathbf{x}) = r$, for all $\mathbf{x} \geq 0$.
- For any production function $g(\mathbf{x})$, a ***local** characterization of returns to scale at point* $\mathbf{x}$ is given by the scale elasticity $SE(\mathbf{x})$ as follows
  SE(x) > 1, = 1, or < 1 corresponds, respectively, to local IRTS, CRTS, or DRTS in the neighborhood of point $\mathbf{x}$.

  This gives the intuitive result: a technology exhibits *local* IRTS, CRTS, or DRTS if a (small) proportional change in all inputs generates, respectively, a more than proportional, proportional, or less than proportional change in output.

9.4.2 From the cost function

Assume that the firm minimizes cost. Substituting the first order necessary condition (2a) into the scale elasticity SE in (11) gives

$$SE(\mathbf{x}) = \sum_i \frac{\partial g(\mathbf{x})}{\partial x_i} \cdot \frac{x_i}{g(\mathbf{x})} = \sum_i \frac{w_i}{\lambda^c} \cdot \frac{x_i^c}{g(\mathbf{x}^c)}, \text{ from (2a),}$$

$$= \frac{\sum_i \frac{w_i \cdot x_i^c}{y}}{\frac{\partial C}{\partial y}}, \text{ from (3b) stating that } \lambda^c = \frac{\partial C}{\partial y},$$

$$= \frac{AC}{MC}, \qquad (12)$$

where AC = $\sum_i \frac{w_i \cdot x_i^c}{y}$ = average cost, and MC = $\frac{\partial C}{\partial y}$ = marginal cost. Expression (12) implies that, under cost minimization, the *scale elasticity SE can be measured at the ratio of average cost to marginal cost.*

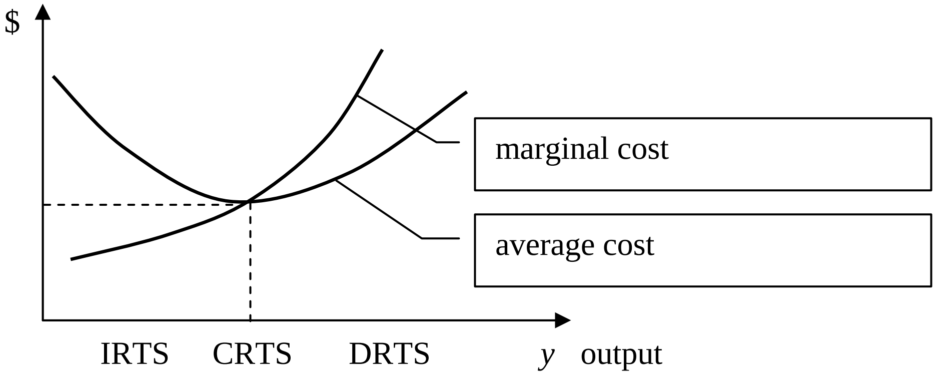
Combining this with our earlier analysis, this gives the following important result:

> *Under cost minimizing behavior,* $\frac{AC}{MC}$ *> 1, = 1, or < 1 corresponds, respectively, to IRTS, CRTS, or DRTS in the neighborhood of point (*$\mathbf{w}$*, y).*

But we have analyzed the relationships between AC and MC earlier. We found that AC(y) being decreasing, constant, or increasing in y implies, respectively, that AC > MC, AC = MC, or AC < MC. In the case where the AC(y) curve has a U-shape, this has the following *intuitive implications* (see figure 7.3):

IRTS corresponds to the region where AC(y) is *decreasing*, with AC > MC,
CRTS corresponds to the region where AC(y) is *constant*, with AC = MC,
DRTS corresponds to the region where AC(y) is *increasing*, with AC < MC.

Figure 9.3



9.5 Input Substitution

9.5.1 Substitutes and complements

Definition: Two inputs *i* and *j* are said to be *substitutes* (*complements*) if $\frac{\partial x_i^c}{\partial w_j} > 0$ (< 0), for $i \neq j$; $i, j = 1, \ldots, n$.

This has the following intuitive interpretation. Consider an increase in the price w_j. Input demand functions being downward sloping ($\frac{\partial x_j^c}{\partial w_j} < 0$), this implies a decrease in x_j^c. When inputs *i* and *j* are *substitutes* ($\frac{\partial x_i^c}{\partial w_j} > 0$), this means that x_i^c increases while x_j^c decreases (i.e., they move in *opposite directions*). Alternatively, when inputs *i* and *j* are *complements* ($\frac{\partial x_i^c}{\partial w_j} < 0$), this means that both x_i^c and x_j^c decrease (i.e., they move in the *same direction*).

Note: We know that $\mathbf{x}^c(y, \mathbf{w})$ satisfies the symmetry restrictions: $\frac{\partial x_i^c}{\partial w_j} = \frac{\partial x_j^c}{\partial w_i}$. Thus, input *i* is a substitute (complement) for input *j* if and only if input *j* is a substitute (complement) for input *i*, $i \neq j$.

The strength of the substitution between inputs *i* and *j* can be measured by the Allen elasticity of substitution.

Definition: The *Allen elasticity of substitution* (AES) between inputs *i* and *j* is defined as

$$\sigma_{ij} = \frac{\partial x_i^c}{\partial w_j} \cdot \frac{C}{x_i^c \cdot x_j^c}, \tag{13a}$$

$i \neq j$; $i, j = 1, \ldots, n$.

Note that the Allen elasticity of substitution (13a) can be alternatively written as

$$\sigma_{ij} = \frac{\partial \ln(x_i^c)}{\partial \ln(w_j)} \cdot \frac{1}{S_j}, \tag{13b}$$

where $\frac{\partial \ln(x_i^c)}{\partial \ln(w_j)} = \frac{\partial x_i^c}{\partial w_j} \cdot \frac{w_j}{x_i^c}$ is the cross price elasticity of x_i^c with respect to w_j, and $S_j = \frac{w_j \cdot x_j^c}{C}$ is the *j*-th cost share.

The Allen elasticity of substitution has two important properties:

- *symmetric*: $\sigma_{ij} = \sigma_{ji}$ (this follows from (13a) and the symmetry restrictions $\frac{\partial x_i^c}{\partial w_j} = \frac{\partial x_j^c}{\partial w_i}$).
- *unit free* (since both the elasticity $\frac{\partial \ln(x_i^c)}{\partial \ln(w_j)}$ and the cost share S_j are unit free in (13b)).

It follows that two inputs i and j are *substitutes* (*complements*) if $\sigma_{ij} = \sigma_{ji} > 0$ (< 0), for $i \neq j$; $i, j = 1, \ldots, n$.

Note: In general, the homogeneity of degree zero of $\mathbf{x}^c(y, \mathbf{w})$ in $\mathbf{w}$ implies (from Euler's theorem) that

$$\sum_i \frac{\partial x_j^c}{\partial w_i} \cdot w_i = 0, \text{ or } \sum_{i \neq j} \frac{\partial x_j^c}{\partial w_i} \cdot w_i = -\frac{\partial x_j^c}{\partial w_j} \cdot w_j \geq 0.$$

This states that a *weighted sum of the cross-price effects* $\frac{\partial x_j^c}{\partial w_i}$ (with $w_i > 0$ as weight) *must be positive*. This has two important implications:

- When n = 2, this implies that $\frac{\partial x_j^c}{\partial w_i} \geq 0$, i.e. that $\sigma_{ij} = \sigma_{ji} \geq 0$, for $i \neq j$. This means that, in the two-input case ($n = 2$), inputs are *necessarily substitutes:* $\sigma_{12} = \sigma_{21} \geq 0$. In other words, in the two-input case, the Allen elasticity of substitution has a lower bound of zero.
- Under cost minimization, *complementary inputs* satisfying $\sigma_{ij} < 0$ for $i \neq j$ can be observed only when there are *at least three inputs* ($n \geq 3$).

9.5.2 CES technology

A CES (or "constant elasticity of substitution") production function is given by

$$y = g(\mathbf{x}) = A \cdot (\alpha_1 \cdot x_1^{\rho} + \alpha_2 \cdot x_2^{\rho} + \ldots + \alpha_n \cdot x_n^{\rho})^{v/\rho}, \tag{14a}$$

where $A > 0$, $\alpha_i > 0$, the α's are normalized such that $\sum_i \alpha_i = 1$, $v > 0$, and $-\infty \leq \rho \leq 1$.

As we will see below,

- v can be interpreted as the "*homogeneity*" parameter
- ρ is a parameter representing the possibilities of *substitution* among inputs. These possibilities of substitution can also be expressed using the *Allen elasticity of substitution*. In the CES case, we will show below that the Allen elasticity of substitution between any two inputs is $\sigma = 1/(1-\rho)$, with $0 \leq \sigma \leq \infty$. Since ρ is treated as a constant parameter, it follows that the CES specification implies *constant Allen elasticities of substitution among inputs* (which gave it its name).

Taking the log, note that the CES specification (14a) can be alternatively expressed as

$$\ln\left(\frac{y}{A}\right) = \frac{v}{\rho} \cdot \ln(\alpha_1 \cdot x_1^{\rho} + \alpha_2 \cdot x_2^{\rho} + \ldots + \alpha_n \cdot x_n^{\rho}), \text{ or}$$

$$\left(\frac{y}{A}\right)^{\rho/v} = (\alpha_1 \cdot x_1^{\rho} + \alpha_2 \cdot x_2^{\rho} + \ldots + \alpha_n \cdot x_n^{\rho}). \tag{14b}$$

Homogeneous technology: For all $t > 0$,

$$g(t \cdot \mathbf{x}) = A \cdot \Big(\sum_i \alpha_i \cdot (t \cdot x_i)^{\rho}\Big)^{\frac{v}{\rho}} = A \cdot \Big(\sum_i \alpha_i \cdot t^{\rho} \cdot x_i^{\rho}\Big)^{\frac{v}{\rho}} = A \cdot t^{v} \cdot \Big(\sum_i \alpha_i \cdot x_i^{\rho}\Big)^{\frac{v}{\rho}}$$

$$= t^{v} \cdot g(\mathbf{x}), \quad \text{for all } t > 0.$$

Thus, the CES production function $y = A \cdot (\alpha_1 \cdot x_1^{\rho} + \alpha_2 \cdot x_2^{\rho} + \ldots + \alpha_n \cdot x_n^{\rho})^{v/\rho}$ is *homogeneous of degree v* in **x** (with scale elasticity $SE = v$).
For example, the CES specification (14a) means a CRTS production technology when $v = SE = 1$.

Cost minimizing input demand functions:
In the CES specification (14a),

$$\tfrac{\partial g}{\partial x_i} = \tfrac{v}{\rho} \cdot A \cdot (\alpha_1 \cdot x_1^{\rho} + \alpha_2 \cdot x_2^{\rho} + \ldots + \alpha_n \cdot x_n^{\rho})^{\frac{v-\rho}{\rho}} \cdot \alpha_i \cdot \rho \cdot x_i^{\rho-1}, \; i = 1, \ldots, n.$$

From (2b), The FONC for cost minimization imply

$$\frac{w_i}{w_j} = \frac{\frac{\partial g}{\partial x_i}}{\frac{\partial g}{\partial x_j}} = \frac{\alpha_i \cdot x_i^{\rho-1}}{\alpha_j \cdot x_j^{\rho-1}}, \text{ for all } i, j.$$

Solving for x_i gives

$$x_i = x_j \cdot (\tfrac{w_i}{w_j} \cdot \tfrac{\alpha_j}{\alpha_i})^{\frac{1}{\rho-1}}, \text{ or } x_i^{\rho} = x_j^{\rho} \cdot (\tfrac{w_i}{w_j} \cdot \tfrac{\alpha_j}{\alpha_i})^{\frac{\rho}{\rho-1}}.$$

Substituting this expression into the production function (14b) gives

$$(\tfrac{y}{A})^{\frac{\rho}{v}} = \sum_i \alpha_i \cdot x_j^{\rho} \cdot (\tfrac{w_i}{w_j} \cdot \tfrac{\alpha_j}{\alpha_i})^{\frac{\rho}{\rho-1}} = x_j^{\rho} \cdot (\tfrac{\alpha_j}{w_j})^{\frac{\rho}{\rho-1}} \cdot \sum_i \alpha_i \cdot (\tfrac{w_i}{\alpha_i})^{\frac{\rho}{\rho-1}} = x_j^{\rho} \cdot (\tfrac{\alpha_j}{w_j})^{\frac{\rho}{\rho-1}} \cdot \sum_i (\alpha_i)^{\frac{1}{1-\rho}} \cdot (w_i)^{\frac{\rho}{\rho-1}}.$$

Solving for x_j yields

$$x_j^{\rho} = (\tfrac{y}{A})^{\frac{\rho}{v}} \cdot (\tfrac{\alpha_j}{w_j})^{\frac{\rho}{1-\rho}} \cdot (\sum_i (\alpha_i)^{\frac{1}{1-\rho}} \cdot (w_i)^{\frac{\rho}{\rho-1}})^{-1},$$

which gives the *cost minimizing input demand function*

$$x_j^c = (\tfrac{y}{A})^{\frac{1}{v}} \cdot (\tfrac{\alpha_j}{w_j})^{\frac{1}{1-\rho}} \cdot (\sum_i (\alpha_i)^{\frac{1}{1-\rho}} \cdot (w_i)^{\frac{\rho}{\rho-1}})^{-\frac{1}{\rho}}, \; j = 1, \ldots, n. \tag{15}$$

Indirect cost function:
The CES indirect cost function is

$$\begin{aligned} C(y, \mathbf{w}) &= \sum_j w_j \cdot x_j^c \\ &= \sum_j w_j \cdot (\tfrac{y}{A})^{\frac{1}{v}} \cdot (\tfrac{\alpha_j}{w_j})^{\frac{1}{1-\rho}} \cdot (\sum_i (\alpha_i)^{\frac{1}{1-\rho}} \cdot (w_i)^{\frac{\rho}{\rho-1}})^{-\frac{1}{\rho}} \\ &= (\tfrac{y}{A})^{\frac{1}{v}} \cdot (\sum_i (\alpha_i)^{\frac{1}{1-\rho}} \cdot (w_i)^{\frac{\rho}{\rho-1}})^{-\frac{1}{\rho}} \cdot \sum_j w_j \cdot (\tfrac{\alpha_j}{w_j})^{\frac{1}{1-\rho}} \\ &= (\tfrac{y}{A})^{\frac{1}{v}} \cdot (\sum_i (\alpha_i)^{\frac{1}{1-\rho}} \cdot (w_i)^{\frac{\rho}{\rho-1}})^{-\frac{1}{\rho}} \cdot \sum_j (\alpha_j)^{\frac{1}{1-\rho}} \cdot (w_j)^{\frac{\rho}{1-\rho}} \\ &= (\tfrac{y}{A})^{\frac{1}{v}} \cdot (\sum_i (\alpha_i)^{\frac{1}{1-\rho}} \cdot (w_i)^{\frac{\rho}{\rho-1}})^{\frac{\rho-1}{\rho}}. \end{aligned} \tag{16}$$

The Allen elasticities of substitution:
From (13a), we have seen that the Allen elasticity of substitution between input *i* and *j*, is

$$\sigma_{ij} = \sigma_{ji} = \tfrac{\partial x_j^c}{\partial w_i} \cdot \tfrac{C}{x_i^c \cdot x_j^c}, \; i \neq j.$$

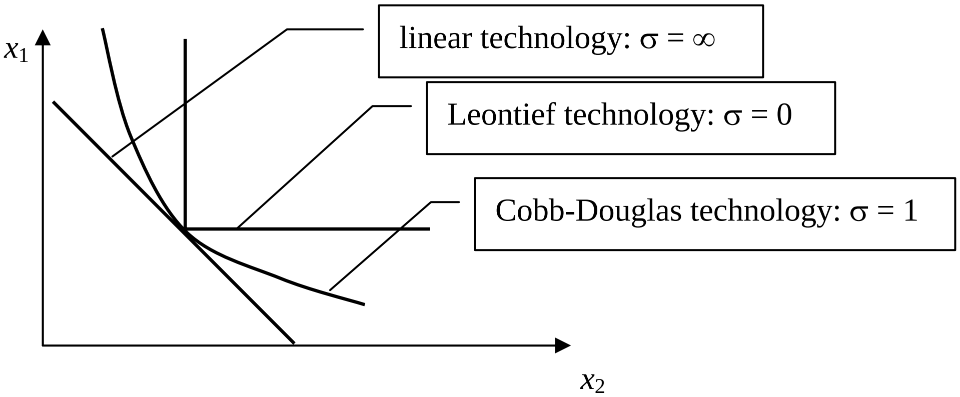
In the CES case, using (15) and (16), this becomes

$$\sigma_{ij} = \sigma_{ji} = \tfrac{1}{1-\rho} \cdot \alpha_i^{\frac{1}{1-\rho}} \cdot w_i^{\frac{1}{\rho-1}} \cdot (\tfrac{\alpha_j}{w_j})^{\frac{1}{\rho-1}} = \tfrac{1}{1-\rho}.$$

This identifies the ρ parameter as measuring the possibilities of *substitution* among inputs. It also shows that the CES specification implies *constant elasticities of substitution* $\sigma_{ij} = \sigma = \frac{1}{1-\rho}$

between *any two inputs i* and *j*, $i \neq j$. Given $-\infty \leq \rho \leq 1$, the Allen elasticity of substitution σ can thus vary between 0 (corresponding to $\rho = -\infty$) and ∞ (corresponding to $\rho = 1$). In other words, in the CES specification, the Allen elasticity of substitution σ is bounded between 0 and ∞: $0 \leq \sigma \leq \infty$. This is illustrated in figure 9.4.

Figure 9.4: shape of isoquants



Leontief technology

The Leontief technology is a special case of the CES specification, where $\alpha_i = a_i^{-\rho}$, $\rho = -\infty$ (corresponding to $\sigma = 0$). With $\sigma = 0$, it corresponds to the *smallest possible amount of substitution among all inputs*.

From (16), the associated indirect cost function is

$$C(y, \mathbf{w}) = \lim_{\rho \to \infty} (\tfrac{y}{A})^{\frac{1}{v}} \cdot (\sum_i a_i^{\frac{-\rho}{1-\rho}} \cdot w_i^{\frac{\rho}{\rho-1}})^{\frac{\rho-1}{\rho}} = (\tfrac{y}{A})^{\frac{1}{v}} \cdot \sum_i a_i \cdot w_i .$$

Using Shephard's lemma, the associated cost minimizing input demand function is

$$x_i^c(y, \mathbf{w}) = \tfrac{\partial C}{\partial w_i} = (\tfrac{y}{A})^{\frac{1}{v}} \cdot a_i, \quad i = 1, \ldots, n. \text{ This implies that } \tfrac{x_i}{a_i} = (\tfrac{y}{A})^{\frac{1}{v}}, \ i = 1, \ldots, n.$$

This corresponds to the Leontief production function (see figure 9.4)

$$y = A \cdot [\min\{\tfrac{x_1}{a_1}, \tfrac{x_2}{a_2}, \ldots, \tfrac{x_n}{a_n}\}]^{v},$$

where the a_i are "*fixed proportion" input-output ratios*. These fixed proportions characterize the Leontief technology, which exhibits *no possibility of substitution among inputs* ($\sigma_{ij} = 0$).

Cobb-Douglas technology

The Cobb-Douglas technology is a special case of the CES specification, when $\rho = 0$ (corresponding to $\sigma = 1$) and $\beta_i = v \cdot \alpha_i$, $i = 1, \ldots, n$. With $\sigma = 1$, it corresponds to a *unitary Allen elasticity of substitution among all inputs*.

From (14), the Cobb-Douglas production function is

$$\ln(y) = \ln(A) + \lim_{\rho \to 0}(\tfrac{v}{\rho} \cdot \ln(\alpha_1 \cdot x_1^{\rho} + \alpha_2 \cdot x_2^{\rho} + \ldots + \alpha_n \cdot x_n^{\rho})),$$

$$= \ln(A) + v \cdot \lim_{\rho \to 0} \frac{\partial \ln(\alpha_1 \cdot x_1^{\rho} + \ldots + \alpha_n \cdot x_n^{\rho})}{\partial \rho}, \text{ using l'Hôpital's rule,}$$

$$= \ln(A) + v \cdot \lim_{\rho \to 0}(\frac{\partial(\alpha_1 \cdot x_1^{\rho} + \ldots + \alpha_n \cdot x_n^{\rho})}{\partial \rho} \cdot \frac{1}{\alpha_1 \cdot x_1^{\rho} + \ldots + \alpha_n \cdot x_n^{\rho}}),$$

$$= \ln(A) + v \cdot \lim_{\rho \to 0}\left(\frac{\alpha_1 \cdot x_1^{\rho} \cdot \ln(x_1) + \ldots + \alpha_n \cdot x_n^{\rho} \cdot \ln(x_n)}{\alpha_1 \cdot x_1^{\rho} + \ldots + \alpha_n \cdot x_n^{\rho}}\right), \text{ since } \tfrac{\partial z^{\rho}}{\partial \rho} = z^{\rho} \cdot \ln(z),$$

$$= \ln(A) + v \cdot \frac{\alpha_1 \cdot \ln(x_1) + \ldots + \alpha_n \cdot \ln(x_n)}{\alpha_1 + \ldots + \alpha_n},$$

$$= \ln(A) + v \cdot \sum_i \alpha_i \cdot \ln(x_i), \text{ given the normalization rule } \sum_i \alpha_i = 1,$$

$$= \ln(A) + \sum_i \beta_i \cdot \ln(x_i), \text{ where } \beta_i = v \cdot \alpha_i,\ i = 1, \ldots, n.$$

which gives the *Cobb-Douglas production function*

$$y = A \cdot x_1^{\beta 1} \cdot x_2^{\beta 2} \cdot \ldots \cdot x_n^{\beta n}, \text{ or } \ln(y) = \ln(A) + v \cdot \sum_i \alpha_i \cdot \ln(x_i).$$

With $\beta_i = v \cdot \alpha_i$, and the normalization rule $\sum_i \alpha_i = 1$, it follows that the Cobb-Douglas technology is *homogeneous of degree* $v = \sum_i \beta_i$.

From (16), the Cobb-Douglas indirect cost function satisfies

$$\ln(C) = \lim_{\rho \to 0}\left(\tfrac{1}{v} \cdot \ln(\tfrac{y}{A}) + (1 - \tfrac{1}{\rho}) \cdot \ln\left(\sum_i \alpha_i^{\frac{1}{1-\rho}} \cdot w_i^{\frac{\rho}{\rho-1}}\right)\right),$$

$$= \tfrac{1}{v} \cdot \ln(\tfrac{y}{A}) - \lim_{\rho \to 0}\left(\ln\left(\sum_i \alpha_i^{\frac{1}{1-\rho}} \cdot w_i^{\frac{\rho}{\rho-1}}\right) \cdot \tfrac{1}{\rho}\right), \text{ since } \sum_i \alpha_i = 1,$$

$$= \tfrac{1}{v} \cdot \ln(\tfrac{y}{A}) - \lim_{\rho \to 0}\left(\frac{\partial \ln\left(\sum_i \alpha_i^{\frac{1}{1-\rho}} \cdot w_i^{\frac{\rho}{\rho-1}}\right)}{\partial \rho}\right), \text{ from L'Hôpital's rule,}$$

$$= \tfrac{1}{v} \cdot \ln(\tfrac{y}{A}) - \lim_{\rho \to 0}\left(\left(\sum_i \alpha_i^{\frac{1}{1-\rho}} \cdot w_i^{\frac{\rho}{\rho-1}} \cdot \ln(\alpha_i) \cdot \tfrac{1}{(1-\rho)^2} - \sum_i \alpha_i^{\frac{1}{1-\rho}} \cdot w_i^{\frac{\rho}{\rho-1}} \cdot \ln(w_i) \cdot \tfrac{1}{(\rho-1)^2}\right) \cdot \frac{1}{\sum_i \alpha_i^{\frac{1}{1-\rho}} \cdot w_i^{\frac{\rho}{\rho-1}}}\right), \text{ since}$$

$$\tfrac{\partial z^{\rho}}{\partial \rho} = z^{\rho} \cdot \ln(z),$$

$$= \tfrac{1}{v} \cdot \ln(\tfrac{y}{A}) - \left(\sum_i \alpha_i \cdot \ln(\alpha_i) - \sum_i \alpha_i \cdot \ln(w_i)\right), \text{ since } \sum_i \alpha_i = 1,$$

$$= \tfrac{1}{v} \cdot \ln(\tfrac{y}{A}) + \sum_i \alpha_i \cdot \ln(\tfrac{w_i}{\alpha_i}),$$

$$= \ln(k) + \tfrac{1}{v} \cdot \ln(y) + \sum_i \alpha_i \cdot \ln(w_i),$$

where $\ln(k) = -\tfrac{1}{v} \cdot \ln(A) - \sum_i \alpha_i \cdot \ln(\alpha_i)$, implying that

$$C(y, \mathbf{w}) = k \cdot y^{1/v} \cdot w_1^{\alpha 1} \cdot \ldots \cdot w_n^{\alpha n}, \text{ or}$$
$$\ln(C) = \ln(k) + \tfrac{1}{v} \cdot \ln(y) + \sum_i \alpha_i \cdot \ln(w_i).$$

Given $\beta_i = v \cdot \alpha_i$ and $\sum_i \beta_i = v$ (from the normalization rule $\sum_i \alpha_i = 1$), the *Cobb-Douglas indirect cost function* is

$$C(y, \mathbf{w}) = k \cdot y^{1/v} \cdot w_1^{(\beta 1)/v} \cdot \ldots \cdot w_n^{(\beta n)/v}, \text{ or}$$
$$\ln(C) = \ln(k) + \tfrac{1}{v} \cdot \ln(y) + \sum_i \tfrac{\beta_i}{v} \cdot \ln(w_i).$$

Linear technology

The linear technology is a special case of the CES specification, when $\rho = 1$ (corresponding to $\sigma = \infty$). With $\sigma = \infty$, it corresponds to the *largest possible amount of substitution among all inputs*.

From (14), the *linear production function* is

$$y = A \cdot (\alpha_1 \cdot x_1 + \alpha_2 \cdot x_2 + \ldots + \alpha_n \cdot x_n)^v, \text{ or}$$

$$(\tfrac{y}{A})^{\frac{1}{v}} = \sum_i \alpha_i \cdot x_i \, .$$

Note that, under a linear technology, the SONC is satisfied, but not the SOSC. This illustrates that the linear production function is quasi-concave, but not strictly quasi-concave.

9.5.3 Flexible functional forms

Definition: A *flexible functional form* is a form that does not impose *a priori* restrictions on the Allen elasticities of substitution σ_{ij} (as defined in (13)).

From our analysis, it follows that *neither the Leontief technology, nor the Cobb-Douglas technology, nor the linear technology are flexible functional forms*. Indeed, they each impose *a priori* restrictions on the Allen elasticity of substitution: $\sigma_{ij} = 0$ in the Leontief case, $\sigma_{ij} = 1$ in the Cobb-Douglas case, and $\sigma_{ij} = \infty$ in the linear case.

What about the CES specification?

- In the two input case ($n = 2$), it is a flexible functional form since it does *not* impose a priori restriction on the Allen elasticity of substitution (recall that $\sigma_{12} = \sigma_{21}$ has zero as a lower bound when $n = 2$).
- However, in the general case where $n \geq 3$, CES is *not* a flexible functional form because it restricts the Allen elasticities of substitution to be *the same* among all inputs: $\sigma_{ij} = \sigma$. For example, with $\sigma \geq 0$, it does *not allow for complementary inputs*.

Examples of flexible functional forms for the indirect cost function: (under a homogeneous technology of degree v)

- *Generalized Leontief* specification:

  $$C(y, \mathbf{w}) = (\tfrac{y}{A})^{\frac{1}{v}} \cdot \sum_i \sum_j \alpha_{ij} \cdot w_i^{\frac{1}{2}} \cdot w_j^{\frac{1}{2}} \, .$$

  This includes the Leontief technology as a special case when $a_{ij} = 0$ for $i \neq j$.

- *Translog* specification:

  $$\ln(C(y, \mathbf{w})) = \ln(k) + \tfrac{1}{v} \cdot \ln(y) + \sum_i \alpha_i \cdot \ln(w_i) + \frac{1}{2} \cdot \sum_i \sum_j \alpha_{ij} \cdot \ln(w_i) \cdot \ln(w_j) \, .$$

  This includes the Cobb-Douglas technology as a special case when $\alpha_{ij} = 0$.

9.6 Output Effects

Definition: The *output elasticity* of the cost minimizing input demand functions are

$$\tfrac{\partial \ln(x_i^c)}{\partial \ln(y)} = \tfrac{\partial x_i^c}{\partial y} \cdot \tfrac{y}{x_i^c}, \; i = 1, \ldots, n.$$

In general, $\tfrac{\partial \ln(x_i^c)}{\partial \ln(y)}$ can be positive or negative.

Differentiating the production function ln(y) = ln(g(**x**)) evaluated at $\mathbf{x}^c(y, \mathbf{w})$ with respect to ln(y) gives, using the chain rule,

$$\sum_i \frac{\partial \ln(g(\mathbf{x}))}{\partial \ln(x_i)} \cdot \frac{\partial \ln(x_i^c)}{\partial \ln(y)} = 1. \tag{17}$$

Equation (17) states that the weighted sum of the output elasticities, $\frac{\partial \ln(x_i^c)}{\partial \ln(y)}$, with $\frac{\partial \ln(g(\mathbf{x}))}{\partial \ln(x_i)} > 0$ as weights, must be equal to one. This implies that, *on average*, the output elasticities $\frac{\partial \ln(x_i^c)}{\partial \ln(y)}$ must be *positive*.

9.6.1 Homothetic technology

A *homothetic technology* is a technology represented by a *homothetic* production function $y = g(\mathbf{x})$, i.e. a production function satisfying

$$y = g(\mathbf{x}) = F(h(\mathbf{x})),$$

where $F(h)$ is a strictly increasing function, and $h(\mathbf{x})$ is a function that is *homogeneous of degree one* in **x**.

Under a homothetic technology, consider the marginal rate of substitution between any two inputs i and j

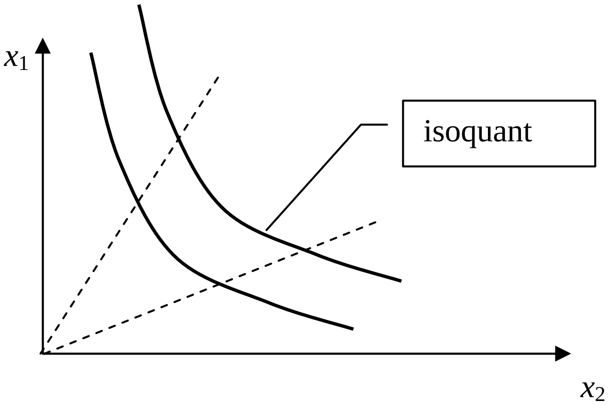
$$\text{MRS}_{ij}(\mathbf{x}) = \frac{\frac{\partial g}{\partial x_i}}{\frac{\partial g}{\partial x_j}} = \frac{\frac{\partial F}{\partial h} \cdot \frac{\partial h}{\partial x_i}}{\frac{\partial F}{\partial h} \cdot \frac{\partial h}{\partial x_j}} = \frac{\frac{\partial h}{\partial x_i}}{\frac{\partial h}{\partial x_j}} = \textit{homogeneous of degree zero in } \mathbf{x},$$

since $h(\mathbf{x})$ is homogeneous of degree one in **x**. Thus, under a homothetic technology,

$$\text{MRS}_{ij}(t \cdot \mathbf{x}) = \text{MRS}_{ij}(\mathbf{x}) \text{ for all } t > 0.$$

This means that, under a *homothetic technology*, the *marginal rate of substitution between any two inputs* (i.e., the slope of isoquants) is *constant along a ray through the origin*. See figure 9.5.

Figure 9.5: Homothetic technology



Denote by G the *inverse function* of F, where $y = F(h)$ is equivalent to $h = G(y)$. Then, cost minimization can be written as

$$C(y, \mathbf{w}) = \underset{\mathbf{x}}{Min} \; \{\mathbf{w} \cdot \mathbf{x} : y = F(h(\mathbf{x})), \mathbf{x} \geq \mathbf{0}, \mathbf{x} \in \mathbf{R}^n\},$$

$$= \underset{\mathbf{x}}{Min} \; \{\mathbf{w} \cdot \mathbf{x} : G(y) = h(\mathbf{x}), \mathbf{x} \geq \mathbf{0}, \mathbf{x} \in \mathbf{R}^n\},$$

$$= \underset{\mathbf{x}}{Min}\ \{\mathbf{w}\cdot t\cdot \mathbf{X} : G(y) = h(t\cdot \mathbf{X}), \mathbf{x} \geq \mathbf{0}, \mathbf{x} \in \mathbf{R}^n\}, \text{ where } \mathbf{x} = t\cdot \mathbf{X},\ t > 0,$$

$$= t\cdot \underset{\mathbf{X}}{Min}\ \{\mathbf{w}\cdot \mathbf{X} : G(y)\cdot \tfrac{1}{t} = h(\mathbf{X}), \mathbf{x} \geq \mathbf{0}, \mathbf{x} \in \mathbf{R}^n\}, \text{ since } h(t\cdot \mathbf{X}) = t\cdot h(\mathbf{X}).$$

Choosing $t = \frac{G(y)}{G(1)}$, it follows that, under a *homothetic technology*, the indirect cost function takes the form

$$C(y, \mathbf{w}) = \frac{G(y)}{G(1)} \cdot \underset{\mathbf{X}}{Min}\ \{\mathbf{w}\cdot \mathbf{X} : G(1) = h(\mathbf{X}), \mathbf{x} \geq \mathbf{0}, \mathbf{x} \in \mathbf{R}^n\} = \frac{G(y)}{G(1)} \cdot C(1, \mathbf{w}).$$

From Shephards' lemma, this implies that

$$\frac{\partial C}{\partial w_i} = x_i^c = G(y)\cdot \frac{\partial C(1,\mathbf{w})}{\partial w_i},\ i = 1, \ldots, n, \text{ or}$$

$$\ln(x_i^c) = \ln(G(y)) + \ln\left(\frac{\partial C(1,\mathbf{w})}{\partial w_i}\right),\ i = 1, \ldots, n.$$

As a result, under a *homothetic technology*, the elasticities of cost minimizing input demand functions are

$$\frac{\partial \ln(x_i^c)}{\partial \ln(y)} = \frac{\partial \ln(G(y))}{\partial \ln(y)} = \text{the same for all inputs, for } i = 1, \ldots, n, \text{ and}$$

$$\frac{\partial \ln(x_i^c)}{\partial \ln(\mathbf{w})} = \frac{\partial \ln\left(\frac{\partial C(1,\mathbf{w})}{\partial w_i}\right)}{\partial \ln(\mathbf{w})} = \text{independent of output } y, \text{ for } i = 1, \ldots, n.$$

This has two important implications: under a *homothetic technology*,

- The *output elasticities* $\frac{\partial \ln(x_i^c)}{\partial \ln(y)}$ are *the same for all inputs*,
- The *price elasticities* $\frac{\partial \ln(x_i^c)}{\partial \ln(\mathbf{w})}$ are *independent of output y*.

From equation (17), it follows that

$$\sum_i \frac{\partial \ln(g(\mathbf{x}))}{\partial \ln(x_i)} \cdot \frac{\partial \ln(G(y))}{\partial \ln(y)} = 1, \text{ or}$$

$$\frac{\partial \ln(x_i^c)}{\partial \ln(y)} = \frac{\partial \ln(G(y))}{\partial \ln(y)} = \left(\sum_i \frac{\partial \ln(g(\mathbf{x}))}{\partial \ln(x_i)}\right)^{-1} = \frac{1}{SE} > 0, \text{ from (11).}$$

Thus, under a *homothetic technology*, the *output elasticities* $\frac{\partial \ln(x_i^c)}{\partial \ln(y)}$ are equal to the *inverse of the scale elasticity SE*.

9.6.2 Homogeneous technology

As a special case of a homothetic technology, consider the situation where $g(\mathbf{x}) = h(\mathbf{x})^r$, and $F(h) = h^r$, where $r > 0$ and $h(\mathbf{x})$ is homogeneous of degree one in $\mathbf{x}$. This corresponds to a *homogeneous technology of order r*, where the production function $g(\mathbf{x}) = h(\mathbf{x})^r$ is homogenous of degree r in $\mathbf{x}$. Then, the inverse function of $y = F(h)$ is $G(y) = y^{1/r}$. It follows that, under a *homogeneous technology of degree r*, the indirect cost function takes the form

$$C(y, \mathbf{w}) = y^{1/r} \cdot C(1, \mathbf{w}), \text{ or}$$

$$\ln(C) = \frac{1}{r} \cdot \ln(y) + \ln(C(1, \mathbf{w})).$$

From Shephard's lemma,

$$x_i^c = y^{\frac{1}{r}} \cdot \frac{\partial C}{\partial w_i (1,\mathbf{w})},\ i = 1, \ldots, n, \text{ or}$$

$$\ln(x_i^c) = \tfrac{1}{r}\cdot\ln(y) + \ln(\tfrac{\partial C}{\partial w_i(1,\mathbf{w})}),\ i = 1, \ldots, n.$$

As a result, under a *homogeneous technology of degree r*, the output elasticity of the cost minimizing input demand functions is

$$\frac{\partial \ln(x_i^c)}{\partial \ln(y)} = \frac{1}{r},\ i = 1, \ldots, n.$$

This shows that, under a *homogeneous technology*, the *output elasticities* $\frac{\partial \ln(x_i^c)}{\partial \ln(y)}$ are equal to the *inverse of the degree of homogeneity r*.

9.7 Long run equilibrium

9.7.1 Industry equilibrium under free entry and exit

Consider a firm facing input prices **w** in an industry exhibiting free entry and exit. Assume that the firm average cost function $\frac{C(y,\mathbf{w})}{y}$ has a U-shape with respect to *y*. Consider the minimization problem

$$\underset{\mathbf{x},y}{Min}\ \{\tfrac{\mathbf{w}\cdot\mathbf{x}}{y} : y = g(\mathbf{x}), \mathbf{x} \geq \mathbf{0}, \mathbf{x} \in \mathbf{R}^n\}. \tag{18a}$$

Denote the solution of this problem by $\mathbf{x}^e(\mathbf{w})$, $y^e(\mathbf{w})$.

The optimization problem (18a) can alternatively written as

$$\underset{\mathbf{x},y}{Min}\ \{\mathbf{w}\cdot\tfrac{\mathbf{x}}{y} : y = g(\mathbf{x}), \mathbf{x} \geq \mathbf{0}, \mathbf{x} \in \mathbf{R}^n\} = \underset{y}{Min}\ \underset{\mathbf{x}}{Min}\ \{\mathbf{w}\cdot\tfrac{\mathbf{x}}{y} : y = g(\mathbf{x}), \mathbf{x} \geq \mathbf{0}, \mathbf{x} \in \mathbf{R}^n\}$$

$$= \underset{y}{Min}\ \{\underset{\mathbf{x}}{Min}\ \{\mathbf{w}\cdot\mathbf{x} : y = g(\mathbf{x}), \mathbf{x} \geq \mathbf{0}, \mathbf{x} \in \mathbf{R}^n\}\cdot\tfrac{1}{y}\} = \underset{y}{Min}\ \{\tfrac{C(y,\mathbf{w})}{y}\},$$

which amounts to choosing $y^e(\mathbf{w})$ so as to minimize the average cost $AC(y) = \frac{C(y,\mathbf{w})}{y}$.

We know that (see figure 9.6)

- $\frac{\partial C}{\partial y} < \frac{C}{y}$ when the average cost function $\frac{C}{y}$ is declining in *y* (corresponding to IRTS).
- $\frac{\partial C}{\partial y} = \frac{C}{y}$ at the minimum of the average cost function $\frac{C}{y}$ with respect to *y* (corresponding to local CRTS).
- $\frac{\partial C}{\partial y} > \frac{C}{y}$ when the average cost function $\frac{C}{y}$ is increasing in *y* (corresponding to DRTS).

It follows that choosing (**x**, *y*) so as to minimize average cost in (18a) involves choosing a firm size y^e that corresponds to CRTS. This point is of special interest for a competitive firm under free entry and exit. When the firm faces output price *p*, we expect the competitive firm to produce at a point where $\frac{\partial C}{\partial y} = p$, i.e. where *marginal cost pricing applies*.

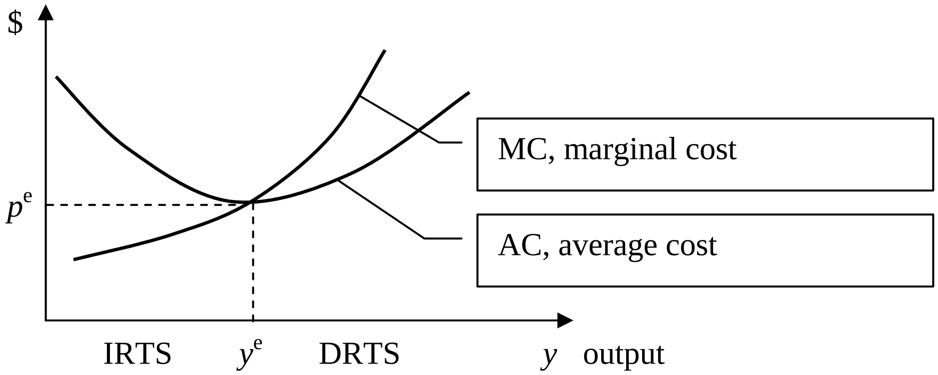
Then, under a U-shape *AC*(*y*) curve, there are three possibilities (see figure 9.6):

- $p = \frac{\partial C}{\partial y} < \frac{C}{y}$. This means that the firm is producing in the region of IRTS. This also means that $p\cdot y < C$, i.e., that profit is negative. Under free exit, this would provide incentives for the firm to exit the industry.
- $p = \frac{\partial C}{\partial y} = \frac{C}{y}$. This means that the firm is producing at the minimum of the average cost function $\frac{C}{y}$ (corresponding to local CRTS). This also means that $p\cdot y = C$, i.e. that profit is zero.

- $p = \frac{\partial C}{\partial y} > \frac{C}{y}$. This means that the firm is producing in the region of DRTS. It also means that $p \cdot y > C$, i.e. that profit is positive. Under free entry, this would provide incentives for similar firms to enter the industry.

Define *long run industry equilibrium* as a situation where there are *no incentives for firms to either enter or exit the industry*. From the above discussion, it follows that *long run industry equilibrium can exist only at a point where* $p = \frac{\partial C}{\partial y} = \frac{C}{y}$*, i.e. where the firm is producing at the minimum of the AC(y) curve, at a point exhibiting local CRTS.*

Figure 9.6



This means that the minimization problem (18a) provides a characterization of long run equilibrium, where both marginal cost pricing ($p = \frac{\partial C}{\partial y}$) and average cost pricing ($p = \frac{C}{y}$) apply. In this context, the *indirect objective function* in (18) can be interpreted as the *long run equilibrium output price*

$$p^e(\mathbf{w}) = \underset{\mathbf{x},y}{Min}\ \{\mathbf{w} \cdot \tfrac{\mathbf{x}}{y} : y = g(\mathbf{x}), \mathbf{x} \geq \mathbf{0}, \mathbf{x} \in \mathbf{R}^n\}. \tag{18b}$$

9.7.2 Properties of the equilibrium

Note that the objective function in (18b) is linear in **w**. For a minimization problem, this implies that the indirect objective function $p^e(\mathbf{w})$ is *concave in* **w**.

Also, applying the envelope theorem to (18b) yields

$$\frac{\partial p^e}{\partial w_i} = \frac{x_i^e(\mathbf{w})}{y^e(\mathbf{w})}, \tag{19a}$$

$i = 1, \ldots, n$. Given $\frac{x_i^e}{y^e} > 0$, this means that *any increase in input price w_i tends to shift up the minimum of the average cost function* and *increase the long run equilibrium output price p^e.*

And, under twice continuous differentiability, $p^e(\mathbf{w})$ being a concave function implies that

$$\frac{\partial^2 p^e}{\partial \mathbf{w}^2} = \frac{\partial(\frac{x^e}{y^e})}{\partial \mathbf{w}} = \text{a symmetric, negative semi-definite matrix.} \tag{19b}$$

This has the following implications:

- The *symmetry* property implies that

$$\frac{\partial(\frac{x_i^e}{y^e})}{\partial w_j} = \frac{\partial(\frac{x_j^e}{y^e})}{\partial w_i}, \text{ for all } i \neq j,$$

i.e. that price effects on relative demand functions ($\frac{\mathbf{x}^e}{y^e}$) are *symmetric*.

- The *negative semi-definiteness* property implies that

$$\frac{\partial(\frac{x_i^e}{y^e})}{\partial w_i} \leq 0, \text{ for } i = 1, \ldots, n,$$

i.e. that the *relative demand function* ($\frac{x_i^e}{y^e}$) is *downward-sloping* with respect to w_i.

9.7.3 Relationships with profit maximization

As discussed above, (18b) is consistent with both marginal cost pricing (corresponding to profit maximization for a competitive firm) as well as average cost pricing (corresponding to long run equilibrium under free entry and exit). This generates the following identities

$$\mathbf{x}^e(\mathbf{w}) = \mathbf{x}^*(p^e(\mathbf{w}), \mathbf{w}), \tag{20a}$$

and

$$y^e(\mathbf{w}) = y^*(p^e(\mathbf{w}), \mathbf{w}), \tag{20b}$$

where $\mathbf{x}^*(p, \mathbf{w})$ and $y^*(p, \mathbf{w})$ are profit maximizing decision rules. Differentiating (20a) and (20b) with respect to $\mathbf{w}$ gives, using the chain rule,

$$\frac{\partial \mathbf{x}^e}{\partial w_i} = \frac{\partial \mathbf{x}^*}{\partial w_i} + \frac{\partial \mathbf{x}^*}{\partial p} \cdot \frac{\partial p^e}{\partial w_i} = \frac{\partial \mathbf{x}^*}{\partial w_i} + \frac{\partial \mathbf{x}^*}{\partial p} \cdot \frac{x_i^e}{y^e}, \; i = 1, \ldots, n,$$

using (19a), and

$$\frac{\partial y^e}{\partial w_i} = \frac{\partial y^*}{\partial w_i} + \frac{\partial y^*}{\partial p} \cdot \frac{\partial p^e}{\partial w_i} = \frac{\partial y^*}{\partial w_i} + \frac{\partial y^*}{\partial p} \cdot \frac{x_i^e}{y^e}, \; i = 1, \ldots, n, \text{ using (19a).}$$

This provides a formal linkage between "*short run" profit maximizing behavior* for a competitive firm (where output price is taken as given) and *long run equilibrium under free entry and exit* (where output price is determined endogenously to be equal to the minimum average cost).