Lecture 8 DUALITY IN FIRM THEORY

8.1 REVIEW
[Profit maximization problem, constrained optimization, cost minimization…]

8.2 Motivation
So far, we have relied on the first order necessary condition (FONC) and the implicit function theorem to investigate the properties of optimal decision rules. While we obtained some general results, they were not always convenient for empirical analysis or for intuitive reasoning. Indeed, solving a system of equations, or inverting matrices, can be tedious and sometimes unintuitive. This suggests the need for some alternative approaches that may be easier to use. Here, we explore the "dual approach" to economic analysis. In this context, the use of the FONC is often called the "primal approach." The dual approach presented below just provides a slightly different way to investigate the general properties of optimal decision rules. The dual approach will also prove useful in benefit-cost analysis.

8.3 The indirect objective function
Consider the constrained optimization problem

$$\underset{\mathbf{x}}{Max}\ \{f(\mathbf{x},\boldsymbol{\alpha}) : \mathbf{h}(\mathbf{x},\boldsymbol{\alpha}) = 0, \mathbf{x} \geq 0, \mathbf{x} \in \mathbf{R}^n\}, \tag{1}$$

where $f(\mathbf{x}, \alpha)$ is the (direct) objective function, $\mathbf{x}$ is a $(n\times1)$ vector of decision variables, α is a $(k\times1)$ vector of "parameters," and $\mathbf{h}(\mathbf{x},\boldsymbol{\alpha}) = \begin{bmatrix} h_1(\mathbf{x},\boldsymbol{\alpha}) \\ \vdots \\ h_m(\mathbf{x},\boldsymbol{\alpha}) \end{bmatrix} = 0$ is a set of m constraints, with $m < n$. Here, the parameters α represent the economic environment: it is the set of all relevant variables that are *not under the control of the decision maker*. Let $\mathbf{x}^*(\alpha)$ denote the solution to this maximization problem. This is a *decision rule* showing how maximizing behavior $\mathbf{x}^*$ adjusts under a changing economic environment α.

Under situation α, define the value of the objective function attained under optimizing behavior by

$$V(\alpha) = f(\mathbf{x}^*(\alpha), \alpha). \tag{2}$$

The function $V(\alpha)$ is called the *indirect objective function*. The indirect objective function $V(\alpha)$ is of interest for two reasons.

- First, it is sometimes subject to direct measurements and thus useful in empirical investigations of economic behavior.
- Second, it provides a direct measure of the effects of the economic environment (as represented by α) on the well being of the decision maker. As such, whether it is directly measurable or not, it gives the foundation of welfare analysis and cost-benefit analysis.

Using the Lagrange approach, consider the *Lagrangean* function $L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\alpha})$ for this problem:

$$\begin{aligned} L(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\alpha}) &= f(\mathbf{x},\boldsymbol{\alpha}) + \boldsymbol{\lambda}^T \cdot \mathbf{h}(\mathbf{x},\boldsymbol{\alpha}) \\ &= f(\mathbf{x},\boldsymbol{\alpha}) + \sum_j \lambda_j \cdot h_j(\mathbf{x},\boldsymbol{\alpha}), \end{aligned}$$

where $\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix}$ is a $(m\times 1)$ vector of Lagrange multipliers, λ_j being associated with the j-th constraint $h_j(\mathbf{x},\boldsymbol{\alpha}) = 0$, $j = 1,\ldots, m$.

Assume that *constraint qualification (CQ) holds*, i.e. that the $m\times n$ matrix $\mathbf{h}_\mathbf{x}(\mathbf{x}^*)$ has n linearly independent columns. Then, from the Lagrange approach, we know that a *necessary condition for* $\mathbf{x}^*$ *to be an interior solution* to the constrained maximization problem is given by the FONC:

$$L_\mathbf{x}(\mathbf{x}^*(\boldsymbol{\alpha}), \boldsymbol{\lambda}^*(\boldsymbol{\alpha}), \boldsymbol{\alpha}) = 0, \tag{3a}$$

$$L_\boldsymbol{\lambda}(\mathbf{x}^*(\boldsymbol{\alpha}), \boldsymbol{\lambda}^*(\boldsymbol{\alpha}), \boldsymbol{\alpha}) = 0. \tag{3b}$$

We have studied how (3a)-(3b) can be solved for $\mathbf{x}^*(\boldsymbol{\alpha})$ and $\boldsymbol{\lambda}^*(\boldsymbol{\alpha})$, or how applying the implicit function theorem to (3a)-(3b) can give insights into the properties of the decision rules $\mathbf{x}^*(\boldsymbol{\alpha})$. Indeed, applying the implicit function theorem to (3a)-(3b) gives the standard comparative static results giving the marginal effects of a change in the parameters $\boldsymbol{\alpha}$ on $\mathbf{x}^*$ and $\boldsymbol{\lambda}^*$:

$$\begin{bmatrix} \mathbf{x}^*_\boldsymbol{\alpha} \\ \boldsymbol{\lambda}^*_\boldsymbol{\alpha} \end{bmatrix} = -\mathbf{H}^{-1} \cdot \begin{bmatrix} L_{\mathbf{x}\boldsymbol{\alpha}}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\alpha}) \\ L_{\boldsymbol{\lambda}\boldsymbol{\alpha}}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\alpha}) \end{bmatrix}, \tag{4}$$

where $\mathbf{H} = \frac{\partial^2 L}{\partial(\mathbf{x},\boldsymbol{\lambda})^2}$ is the bordered Hessian (evaluated at $\mathbf{x}^*(\boldsymbol{\alpha})$ and $\boldsymbol{\lambda}^*(\boldsymbol{\alpha})$) satisfying $\det(\mathbf{H}) \neq 0$ from the SOSC.

Below, we explore the "dual approach," where we investigate these properties starting from the indirect objective function $V(\boldsymbol{\alpha})$.

Note: This analysis is based on the constrained maximization problem (1). However, it can be easily adapted to a *minimization problem*. Indeed, consider the constrained minimization problem

$$V(\boldsymbol{\alpha}) = \underset{\mathbf{x}}{Min} \ \{f(\mathbf{x},\boldsymbol{\alpha}) : \mathbf{h}(\mathbf{x},\boldsymbol{\alpha}) = 0, \mathbf{x} \geq 0, \mathbf{x} \in \mathbf{R}^n\}.$$

It can be written as the *equivalent maximization problem*

$$V(\boldsymbol{\alpha}) = -\underset{\mathbf{x}}{Max} \ \{-f(\mathbf{x},\boldsymbol{\alpha}) : -\mathbf{h}(\mathbf{x},\boldsymbol{\alpha}) = 0, \mathbf{x} \geq 0, \mathbf{x} \in \mathbf{R}^n\},$$

with corresponding Lagrangean

$$L(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\alpha}) = -f(\mathbf{x},\boldsymbol{\alpha}) - \boldsymbol{\lambda}^T \cdot \mathbf{h}(\mathbf{x},\boldsymbol{\alpha}).$$

This shows that the analysis presented below can be adapted to a constrained minimization problem by making two changes: 1) by replacing f by $-f$, and $\mathbf{h}$ by

– **h**; and 2) by replacing V by $–V$. Note that, while this does not affect the FONCs (3a)-(3b), it does *change the sign of the second derivatives of L and V.*

8.3.1 The envelope theorem

Note that the optimum $\mathbf{x}^*(\alpha)$ is necessarily feasible. This means that the constraints $\mathbf{h}(\mathbf{x}, \alpha) = 0$ are necessarily satisfied at $\mathbf{x}^*(\alpha)$, implying that

$$\mathbf{h}(\mathbf{x}^*(\alpha), \alpha) = 0 \text{ for all } \alpha. \quad (5)$$

This has two implications.

- First, it means that

  $$L(\mathbf{x}^*(\alpha), \lambda^*(\alpha), \alpha) \equiv f(\mathbf{x}^*(\alpha), \alpha) + \lambda^*(\alpha)^T \mathbf{h}(\mathbf{x}^*(\alpha), \alpha) = f(\mathbf{x}^*(\alpha), \alpha) = V(\alpha), \quad (6)$$

  from the definition of the indirect objective function in (2).
  This shows that the *Lagrangean evaluated at* $(\mathbf{x}^*(\alpha), \lambda^*(\alpha))$ *is equal to the indirect objective function* $V(\alpha)$.
- Second, in the case where $h(\mathbf{x}, \alpha)$ and $\mathbf{x}^*(\alpha)$ are differentiable functions, differentiating (5) with respect to α (using the chain rule) gives

  $$h_{\mathbf{x}}(\mathbf{x}^*)\, \mathbf{x}_\alpha^* + h_\alpha(\mathbf{x}^*) = 0. \quad (7)$$

  Equation (7) imposes restrictions on the properties of the decision rules $\mathbf{x}_\alpha^* \equiv \frac{\partial \mathbf{x}^*}{\partial \alpha}$.
  Such restrictions must be satisfied for the decision rules $\mathbf{x}^*(\alpha)$ to remain feasible. We will use these properties below.

The envelope theorem: Assume that the indirect objective function is differentiable, that the CQ holds, and that $\mathbf{x}^*$ is an interior solution. Then,

$$\frac{\partial V}{\partial \alpha} \equiv V_\alpha = L_\alpha(\mathbf{x}^*(\alpha), \lambda^*(\alpha), \alpha). \quad (8)$$

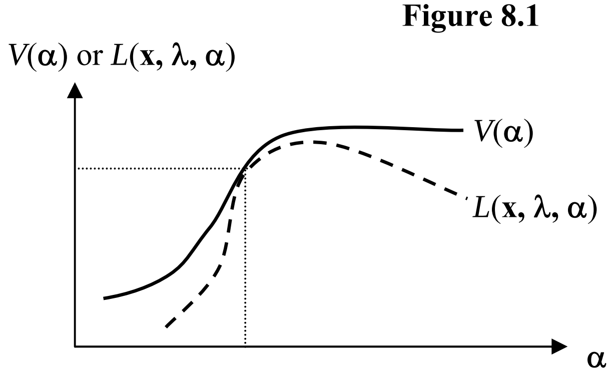
Proof: Differentiating (6) with respect to α gives, using the chain rule,

$$V_\alpha = L_\alpha(\mathbf{x}^*, \lambda^*, \alpha) + L_{\mathbf{x}}(\mathbf{x}^*, \lambda^*, \alpha)\, \mathbf{x}_\alpha^* + L_\lambda(\mathbf{x}^*, \lambda^*, \alpha)\, \lambda_\alpha^*.$$

But, under CQ and an interior solution, the FONC (3a)-(3b) must hold. Substituting (3a)-(3b) into the above expression gives the desired result.

The envelope theorem states that, under some regularity conditions, the direct objective function $L(\mathbf{x}, \lambda, \alpha)$ and the indirect objective function $V(\alpha)$ are tangent with respect to α at $(\mathbf{x}^*(\alpha), \lambda^*(\alpha))$, as they have the same slope with respect to α. See Figure 8.1.

Figure 8.1



8.3.2 Interpreting Lagrange multipliers

Consider the case where $f = f(\mathbf{x})$ and $\mathbf{h}(\mathbf{x},\boldsymbol{\alpha}) = \boldsymbol{\alpha} - \mathbf{g}(\mathbf{x}) = \begin{bmatrix} \alpha_1 - g_1(\mathbf{x}) \\ \vdots \\ \alpha_m - g_m(\mathbf{x}) \end{bmatrix}$ is a $(m\times1)$ vector,

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbf{R}^m$. This means that the parameters $\boldsymbol{\alpha}$ act as "constraint shifters." The corresponding Lagrangean function is

$$L(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\alpha}) = f(\mathbf{x}) + \boldsymbol{\lambda}^T \cdot (\boldsymbol{\alpha} - \mathbf{g}(\mathbf{x})) = f(\mathbf{x}) + \sum_j \lambda_j (\alpha_j - g_j(\mathbf{x})).$$

Assume that $\mathbf{x}^*(\boldsymbol{\alpha})$ is an interior solution, and that $V(\boldsymbol{\alpha})$ is differentiable. Applying the envelope theorem to this problem gives

$$V_{\boldsymbol{\alpha}} = L_{\boldsymbol{\alpha}}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\alpha}) = (\boldsymbol{\lambda}^*)^T,$$

or

$$\tfrac{\partial V}{\partial \alpha_j} = \lambda_j^*,\ j = 1, \ldots, m. \tag{9}$$

This is an important result. It shows that the *Lagrange multipliers* λ_j* *can be interpreted as measuring the marginal effect on the objective function of relaxing the j-th constraint*. In other words, it is the *shadow value of the j-th constraint*.

8.3.3 The primal-dual approach

<u>Alternative characterizations of SOSC</u>:

Recall that, for a maximization problem, the SOSC involves the negative definiteness of the matrix

$$[\mathbf{s}_b(\mathbf{x}_b^*)^T, \mathbf{I}_{n-m}] \cdot L_{\mathbf{xx}}(\mathbf{x}^*,\boldsymbol{\lambda}^*) \cdot \begin{bmatrix} \mathbf{s}_b(\mathbf{x}_b^*) \\ \mathbf{I}_{n-m} \end{bmatrix},$$

where $\mathbf{x} = (\mathbf{x}_a, \mathbf{x}_b)$ with $\mathbf{x}_a = (\mathbf{x}_1, \ldots, \mathbf{x}_m)$, $\mathbf{x}_b = (\mathbf{x}_{m+1}, \ldots, \mathbf{x}_n)$, and $\mathbf{s}_b = -[\mathbf{h}_a]^{-1} \mathbf{h}_b$, with $\mathbf{h}_a \equiv \frac{\partial \mathbf{h}}{\partial \mathbf{x}_a(\mathbf{x}^*)}$ being a $(m\times m)$ matrix, and $\mathbf{h}_b \equiv \frac{\partial \mathbf{h}}{\partial \mathbf{x}_b(\mathbf{x}^*)}$ being a $m\times(n-m)$ matrix.

Thus, SOSC can be expressed as

$$\mathbf{v}_b^T \cdot [-\mathbf{h}_b^T(\mathbf{h}_a^T)^{-1}, \mathbf{I}_{n-m}] \cdot L_{\mathbf{xx}}(\mathbf{x}^*,\boldsymbol{\lambda}^*) \cdot \begin{bmatrix} -(\mathbf{h}_a^T)^{-1}\mathbf{h}_b \\ \mathbf{I}_{n-m} \end{bmatrix} \cdot \mathbf{v}_b < 0 \text{ for all } \mathbf{v}_b \neq 0,$$

where $\mathbf{v}_b$ is a $(n-m)\times1$ vector.

Define the $(m\times1)$ vector $\mathbf{v}_a \equiv -(\mathbf{h}_a)^{-1} \mathbf{h}_b \mathbf{v}_b$ (which can be written as $\mathbf{h}_a\mathbf{v}_a + \mathbf{h}_b\mathbf{v}_b = 0$, or $\mathbf{h}_{\mathbf{x}} \cdot \begin{bmatrix} \mathbf{v}_a \\ \mathbf{v}_b \end{bmatrix} = 0$). Then, SOSC becomes

$$[\mathbf{v}_a^T, \mathbf{v}_b^T] \cdot L_{\mathbf{xx}}(\mathbf{x}^*,\boldsymbol{\lambda}^*) \cdot \begin{bmatrix} \mathbf{v}_a \\ \mathbf{v}_b \end{bmatrix} < 0 \text{ for all } \mathbf{v}_b \neq 0 \text{ satisfying } \mathbf{h}_a \mathbf{v}_a + \mathbf{h}_b \mathbf{v}_b = 0. \tag{10a}$$

Using the $(n+m)\times(n+m)$ *bordered Hessian* $\mathbf{H} \equiv \dfrac{\partial^2 L}{\partial(\mathbf{x},\boldsymbol{\lambda})^2} \equiv \begin{bmatrix} L_{\mathbf{xx}} & (\mathbf{h}_{\mathbf{x}})^T \\ \mathbf{h}_{\mathbf{x}} & 0 \end{bmatrix}$

(evaluated at $\mathbf{x}^*(\boldsymbol{\alpha})$ and $\boldsymbol{\lambda}^*(\boldsymbol{\alpha})$), the SOSC (10a) can be alternatively written as

$$[\mathbf{v}_a^T, \mathbf{v}_b^T, \mathbf{v}_c^T] \cdot \mathbf{H} \cdot \begin{bmatrix} \mathbf{v}_a \\ \mathbf{v}_b \\ \mathbf{v}_c \end{bmatrix} < 0 \text{ for all } \mathbf{v} = \begin{bmatrix} \mathbf{v}_a \\ \mathbf{v}_b \\ \mathbf{v}_c \end{bmatrix} \text{ such that } \mathbf{v}_b \neq 0 \text{ and}$$

$$\mathbf{h}_\mathbf{x} \cdot \begin{bmatrix} \mathbf{v}_a \\ \mathbf{v}_b \end{bmatrix} = 0, \tag{10b}$$

where $\mathbf{v}_c$ is a ($m\times1$) vector. This alternative expression for SOSC can be interpreted in terms of the "negative definiteness of the bordered Hessian **H**, *subject to constraint*, $\mathbf{h}_\mathbf{x} \cdot \begin{bmatrix} \mathbf{v}_a \\ \mathbf{v}_b \end{bmatrix} = 0$," where $\mathbf{h}_\mathbf{x} = (\mathbf{h}_a, \mathbf{h}_b)$ is a ($m\times n$) vector.

The characterization of the SOSC given in (10b) will be useful to obtain the following important result:

<u>Primal-dual results</u>: Assume that f and **h** are twice continuously differentiable, that the constraint qualification (CQ) holds, and that that $\mathbf{x}^*(\boldsymbol{\alpha})$ is interior solution to the maximization problem (1). Then,

- $V_{\boldsymbol{\alpha\alpha}} - L_{\boldsymbol{\alpha\alpha}}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\alpha}) = [L_{\boldsymbol{\alpha}\mathbf{x}}(\mathbf{x}^*, \boldsymbol{\lambda}^*), L_{\boldsymbol{\alpha\lambda}}(\mathbf{x}^*, \boldsymbol{\lambda}^*)] \cdot \begin{bmatrix} \mathbf{x}_{\boldsymbol{\alpha}}^* \\ \boldsymbol{\lambda}_{\boldsymbol{\alpha}}^* \end{bmatrix}$, (11a)
- $\mathbf{u}^T \cdot [V_{\boldsymbol{\alpha\alpha}} - L_{\boldsymbol{\alpha\alpha}}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\alpha})] \cdot \mathbf{u} = \mathbf{u}^T \cdot [L_{\boldsymbol{\alpha}\mathbf{x}}(\mathbf{x}^*, \boldsymbol{\lambda}^*), L_{\boldsymbol{\alpha\lambda}}(\mathbf{x}^*, \boldsymbol{\lambda}^*)] \cdot \begin{bmatrix} \mathbf{x}_{\boldsymbol{\alpha}}^* \\ \boldsymbol{\lambda}_{\boldsymbol{\alpha}}^* \end{bmatrix} \cdot \mathbf{u} \geq 0$ for all ($k\times1$) vector **u** satisfying $\mathbf{h}_\alpha(\mathbf{x}^*, \alpha)\, \mathbf{u} = 0$. (11b)

<u>Proof</u>: Differentiate equation (8) (obtained from the envelope theorem). Using the chain rule, this yields

$$V_{\boldsymbol{\alpha\alpha}} = L_{\boldsymbol{\alpha\alpha}}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\alpha}) + L_{\boldsymbol{\alpha}\mathbf{x}}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \cdot \mathbf{x}_{\boldsymbol{\alpha}}^* + L_{\boldsymbol{\alpha\lambda}}(\mathbf{x}^*, \boldsymbol{\lambda}^*)] \cdot \boldsymbol{\lambda}_{\boldsymbol{\alpha}}^*.$$

This gives equation (11a).

Under twice-continuous differentiability of $L(\mathbf{x}, \lambda, \alpha)$ and $V(\alpha)$, the left-hand side of equation (11a) is symmetric (from Young theorem). It follows that the right-hand side of (11a) is also symmetric.

To prove the positive semi-definiteness, choose $\begin{bmatrix} \mathbf{v}_a \\ \mathbf{v}_b \\ \mathbf{v}_c \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{\boldsymbol{\alpha}}^* \\ \boldsymbol{\lambda}_{\boldsymbol{\alpha}}^* \end{bmatrix} \cdot \mathbf{u}$ in (10b), where **u** is a ($k\times1$) vector. Note that $\mathbf{h}_\mathbf{x} \cdot \begin{bmatrix} \mathbf{v}_a \\ \mathbf{v}_b \end{bmatrix} = 0$ then implies that $\mathbf{h}_\mathbf{x} \cdot \mathbf{x}_\alpha^* \cdot \mathbf{u} = 0$. Using equation (7), this implies that $\mathbf{h}_\alpha(\mathbf{x}^*, \alpha) \cdot \mathbf{u} = 0$. It follows that (10b) gives

$$\mathbf{u}^T \cdot [(\mathbf{x}_{\boldsymbol{\alpha}}^*)^T, (\boldsymbol{\lambda}_{\boldsymbol{\alpha}}^*)^T] \cdot \mathbf{H} \cdot \begin{bmatrix} \mathbf{x}_{\boldsymbol{\alpha}}^* \\ \boldsymbol{\lambda}_{\boldsymbol{\alpha}}^* \end{bmatrix} \cdot \mathbf{u} \leq 0 \text{ for all } \mathbf{u} \text{ satisfying } \mathbf{h}_\alpha \cdot \mathbf{u} = 0.$$

Noting that $[(\mathbf{x}_{\boldsymbol\alpha}^*)^T, (\boldsymbol\lambda_{\boldsymbol\alpha}^*)^T]\cdot \mathbf{H} = -[L_{\boldsymbol{\alpha}\mathbf{x}}(\mathbf{x}^*, \boldsymbol\lambda^*), L_{\boldsymbol{\alpha}\boldsymbol\lambda}(\mathbf{x}^*, \boldsymbol\lambda^*)]$ from the comparative static results (4), this yields

$$\mathbf{u}^T \cdot [L_{\boldsymbol{\alpha}\mathbf{x}}(\mathbf{x}^*, \boldsymbol\lambda^*), L_{\boldsymbol{\alpha}\boldsymbol\lambda}(\mathbf{x}^*, \boldsymbol\lambda^*)]\cdot \begin{bmatrix} \mathbf{x}_{\boldsymbol\alpha}^* \\ \boldsymbol\lambda_{\boldsymbol\alpha}^* \end{bmatrix} \cdot \mathbf{u} \geq 0 \text{ for all } \mathbf{u} \text{ such that } \mathbf{h}_\alpha \cdot \mathbf{u} = 0,$$

which gives (11b).

Equation (11a) states that the matrix $[V_{\boldsymbol{\alpha\alpha}} - L_{\boldsymbol{\alpha\alpha}}(\mathbf{x}^*, \boldsymbol\lambda^*, \boldsymbol\alpha)]$ is equal to the matrix $[L_{\boldsymbol{\alpha}\mathbf{x}}(\mathbf{x}^*, \boldsymbol\lambda^*), L_{\boldsymbol{\alpha}\boldsymbol\lambda}(\mathbf{x}^*, \boldsymbol\lambda^*)]\cdot \begin{bmatrix} \mathbf{x}_{\boldsymbol\alpha}^* \\ \boldsymbol\lambda_{\boldsymbol\alpha}^* \end{bmatrix}$. And equation (11b) states that these two matrices are each *symmetric and positive semi-definite subject to constraint:* $\mathbf{h}_\alpha \cdot \mathbf{u} = 0$. These general results are useful in several ways. First, they are simple in the sense that they do not involve solving a system of equations, or inverting matrices. Second, they imply restrictions on the properties of the decision rules $\mathbf{x}^*(\alpha)$ (as expressed by the derivative $\mathbf{x}_{\boldsymbol\alpha}^* = \frac{\partial \mathbf{x}^*}{\partial \boldsymbol\alpha}$). These restrictions are implied by the maximization hypothesis (1). As illustrated below, this provides a convenient way of obtaining the behavioral implications of economic rationality.

8.3.4 Some special cases

Consider the special case

$$V(\boldsymbol\alpha) = \underset{\mathbf{x}}{Max}\ \{\boldsymbol\alpha^T \cdot \mathbf{x} : \mathbf{h}(\mathbf{x}) = 0, \mathbf{x} \geq 0, \mathbf{x} \in \mathbf{R}^n\}, \tag{1'}$$

where the objective function $f(\mathbf{x}, \alpha) = \alpha^T \cdot \mathbf{x}$ is linear in α, the constraints are $\mathbf{h}(\mathbf{x}) = 0$, and $\alpha \in R^n$ (with $k = n$). Then, the associated Lagrangean is

$$L(\mathbf{x}, \lambda, \alpha) = \alpha^T \cdot \mathbf{x} + \lambda^T \cdot \mathbf{h}(\mathbf{x}).$$

The indirect objective function is

$$V(\alpha) = \alpha^T \cdot \mathbf{x}^*(\alpha) = L(\mathbf{x}^*(\alpha), \lambda^*(\alpha), \alpha).$$

Assume that $\mathbf{x}^*(\alpha)$ is an interior solution, and that $V(\alpha)$ is differentiable. Then, from the *envelope theorem* given in equation (8),

$$\frac{\partial V(\boldsymbol\alpha)}{\partial \boldsymbol\alpha} = \mathbf{x}^*(\boldsymbol\alpha),$$

or

$$\frac{\partial V(\boldsymbol\alpha)}{\partial \alpha_i} = x_i^*, \ i = 1, \ldots, n.$$

This is a very convenient result. It states that, in this special case, *the optimal decision rules can be obtained simply by taking the first derivative of the indirect objective function with respect to* α.

Note that optimal $\mathbf{x}^*$ in (1') are unaffected by a proportional change in α: $\mathbf{x}^*(\alpha) = \mathbf{x}^*(k\,\alpha)$ for all $k > 0$. This implies that $\mathbf{x}^*(\alpha)$ *is homogeneous of degree* ***zero*** *in* α. It follows that

$V(k\,\boldsymbol{\alpha}) = k\,\boldsymbol{\alpha}^T\,\mathbf{x}^*(k\,\boldsymbol{\alpha}) = k\,\boldsymbol{\alpha}^T\,\mathbf{x}^*(\boldsymbol{\alpha}) = k\,V(\boldsymbol{\alpha})$. This implies that the *indirect objective function $V(\boldsymbol{\alpha})$ is homogeneous of degree* ***one*** *in* $\boldsymbol{\alpha}$. When $V(\boldsymbol{\alpha})$ is twice continuously differentiable, this means that

- $\sum_i \frac{\partial V}{\partial \alpha_i}\alpha_i = V$ (from Euler theorem)
- $\frac{\partial V}{\partial \alpha_i}$ (= $x_i^*(\boldsymbol{\alpha})$ from the envelope theorem) is homogeneous of degree zero in $\boldsymbol{\alpha}$
- $\sum_j \frac{\partial x_i^*}{\partial \alpha_j}\alpha_j = 0$, $i = 1, \ldots, n$ (from Euler theorem).

Also, under (1'), $L_{\boldsymbol{\alpha}\mathbf{x}} = \mathbf{I}_n$ and $L_{\boldsymbol{\alpha}\boldsymbol{\lambda}} = 0$. Then, the *primal-dual results* given in (11a)-(11b) are

$$\frac{\partial^2 V}{\partial \boldsymbol{\alpha}^2} \equiv V_{\boldsymbol{\alpha}\boldsymbol{\alpha}} = \mathbf{x}^*_{\boldsymbol{\alpha}}\text{, which is a } (n\times n) \text{ symmetric, positive semi-definite matrix.}$$

When $L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\alpha}) = \boldsymbol{\alpha}^T\,\mathbf{x} + \boldsymbol{\lambda}^T\,\mathbf{h}(\mathbf{x})$, this has two important implications

- $V(\boldsymbol{\alpha})$ is a convex function of $\boldsymbol{\alpha}$ (since $V_{\boldsymbol{\alpha}\boldsymbol{\alpha}}$ is a positive semi-definite matrix for all $\boldsymbol{\alpha}$).
- $\mathbf{x}_{\boldsymbol{\alpha}}$ = a $(n\times n)$ symmetric, positive semi-definite matrix.

This shows that the *indirect objective function $V(\boldsymbol{\alpha})$ is homogeneous of degree one and convex in $\boldsymbol{\alpha}$, that $\frac{\partial V}{\partial \boldsymbol{\alpha}} = \mathbf{x}^*(\boldsymbol{\alpha})$ is homogeneous of degree zero in $\boldsymbol{\alpha}$, and that the symmetry, positive semi-definiteness of the matrix $\mathbf{x}_{\boldsymbol{\alpha}}^*$ are general implications of the maximization hypothesis (1').*

Note: The *convexity* of the indirect objective function is a *general property* of maximization problems where the *objective function is linear in* $\boldsymbol{\alpha}$. To see that, consider the more general problem

$$V(\boldsymbol{\alpha}) = \underset{\mathbf{x}}{Max}\ \{\boldsymbol{\alpha}^T \cdot \mathbf{x} : \mathbf{x} \in \mathbf{F}\}$$

where $\mathbf{F}$ is the feasible set for $\mathbf{x}$.

Then *$V(\boldsymbol{\alpha})$ is convex in $\boldsymbol{\alpha}$, even without interior solution or differentiability*.

Proof: Consider any two situations $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_1$. Define $\boldsymbol{\alpha}_2 = \theta\,\boldsymbol{\alpha}_0 + (1-\theta)\,\boldsymbol{\alpha}_1$, for any θ, $0 \le \theta \le 1$. By definition of a maximum, we have

$$V(\boldsymbol{\alpha}_0) \equiv \boldsymbol{\alpha}_0^T\,\mathbf{x}^*(\boldsymbol{\alpha}_0) \ge \boldsymbol{\alpha}_0^T\,\mathbf{x}^*(\boldsymbol{\alpha}_2),$$

and

$$V(\boldsymbol{\alpha}_1) \equiv \boldsymbol{\alpha}_1^T\,\mathbf{x}^*(\boldsymbol{\alpha}_1) \ge \boldsymbol{\alpha}_1^T\,\mathbf{x}^*(\boldsymbol{\alpha}_2).$$

Multiplying the first expression by θ, the second by $(1-\theta)$, and summing the two gives $\theta\cdot V(\boldsymbol{\alpha}_0) + (1-\theta)\cdot V(\boldsymbol{\alpha}_1) \ge (\theta\cdot\boldsymbol{\alpha}_0 + (1-\theta)\cdot\boldsymbol{\alpha}_1)^T\cdot\mathbf{x}^*(\boldsymbol{\alpha}_2) \equiv V(\theta\cdot\boldsymbol{\alpha}_0 + (1-\theta)\cdot\boldsymbol{\alpha}_1)$, which proves that $V(\boldsymbol{\alpha})$ is a convex function of $\boldsymbol{\alpha}$.

8.4 Profit maximization

When $\mathbf{x}$ denotes a *netput* vector (where output quantities are defined to be positive and input quantities negative), $\boldsymbol{\alpha}$ is the *price vector* for $\mathbf{x}$, then $\boldsymbol{\alpha}^T\,\mathbf{x}$ denotes *profit*, and (1') represents a *standard profit maximization problem, where V(*α*) is the* ***indirect profit function***.

From the above analysis, we obtain:

- The indirect profit function $V(\alpha)$ is *homogeneous of degree one and convex* in prices α (implying that $\sum_i \frac{\partial V}{\partial \alpha_i}\alpha_i = V$ under differentiability).
- Under differentiability and an interior solution, from the envelope theorem (8),
  $$\frac{\partial V}{\partial \boldsymbol{\alpha}} = \mathbf{x}^*(\boldsymbol{\alpha}),$$
  or
  $$\frac{\partial V}{\partial \alpha_i} = x_i^*(\boldsymbol{\alpha}),\ i = 1, \ldots, n.$$
  This is called *Hotelling lemma* in production theory: *the derivative of the indirect profit function V(*α*) with respect to prices* α *gives the profit maximizing netput decision rules* $\mathbf{x}^*(\boldsymbol{\alpha})$.
- $\frac{\partial V}{\partial \boldsymbol{\alpha}} = \mathbf{x}^*(\boldsymbol{\alpha})$ is *homogeneous of degree zero in prices* α (implying that $\sum_j \frac{\partial x_i^*}{\partial \alpha_j}\alpha_j =$ 0, $i = 1, \ldots, n$, under differentiability)
- From the primal-dual results (11a)-(11b)
  $\mathbf{x}_{\alpha}^*$ = a $(n \times n)$ *symmetric, positive semi-definite matrix*,
  which implies that *profit maximizing output supply (input demand) functions are upward (downward) sloping*, $\frac{\partial x_i^*}{\partial \alpha_i} \geq 0$, and that *price effects satisfy the symmetry property*: $\frac{\partial x_i^*}{\partial \alpha_j} = \frac{\partial x_j^*}{\partial \alpha_i}$, for all $i \neq j$.

As we have seen earlier using a "primal approach," these properties are implied by profit maximizing behavior. This illustrates the usefulness of the dual approach: from the envelope theorem and the primal-dual results, the optimal decision rules and their properties can be obtained and analyzed with relative ease.

8.5 Cost minimization

When $\mathbf{x}$ denotes a vector of *inputs*, α is the *price vector* for $\mathbf{x}$, then $\alpha^T\,\mathbf{x}$ denotes *cost*. Consider the following optimization problem

$$V(\boldsymbol{\alpha}) = \underset{\mathbf{x}}{Min}\ \{\boldsymbol{\alpha}^T \cdot \mathbf{x} : \mathbf{h}(\mathbf{x}) = 0, \mathbf{x} \geq 0, \mathbf{x} \in \mathbf{R}^n\},$$

which has for solution $\mathbf{x}^c$. It represents a *standard cost minimization problem, where V(*α*) is the* ***indirect cost function***. It can be written equivalently as

$$V(\boldsymbol{\alpha}) = -\underset{\mathbf{x}}{Max}\ \{-\boldsymbol{\alpha}^T \cdot \mathbf{x} : \mathbf{h}(\mathbf{x}) = 0, \mathbf{x} \geq 0, \mathbf{x} \in \mathbf{R}^n\}. \qquad (1")$$

From the above analysis, we obtain:

- The indirect cost function $V(\alpha)$ is *homogeneous of degree one and concave in prices* α (implying that $\sum_i \frac{\partial V}{\partial \alpha_i}\alpha_i = V$ under differentiability).
- Under differentiability and an interior solution, from the envelope theorem (8),

$$\frac{\partial V}{\partial \boldsymbol{\alpha}} = \mathbf{x}^c(\boldsymbol{\alpha}),$$

or

$$\frac{\partial V}{\partial \alpha_i} = x_i^c(\boldsymbol{\alpha}),\ i = 1, \ldots, n.$$

This is called *Shephard lemma* in production theory: *the derivative of the indirect cost function V(α) with respect to prices α gives the cost minimizing input demand functions* $\mathbf{x}^c(\alpha)$.

- $\frac{\partial V}{\partial \boldsymbol{\alpha}} = \mathbf{x}^c(\boldsymbol{\alpha})$ is *homogeneous of degree zero in prices* α (implying that $\sum_j \frac{\partial x_i^c}{\partial \alpha_j} \alpha_j = 0$, $i = 1, \ldots, n$, under differentiability)
- From the primal-dual results (11a)-(11b)

  $\mathbf{x}_{\boldsymbol{\alpha}}{}^{c}$ = a $(n\times n)$ *symmetric, negative semi-definite matrix*,

  which implies that *cost minimizing input demand functions are downward sloping*, $\frac{\partial x_i^c}{\partial \alpha_i} \leq 0$, and that *price effects satisfy the symmetry property*: $\frac{\partial x_i^c}{\partial \alpha_j} = \frac{\partial x_j^c}{\partial \alpha_i}$, for all $i \neq j$.

As we have seen earlier using a "primal approach," these properties are implied by cost minimizing behavior. Again, this illustrates the usefulness of the dual approach: from the envelope theorem and the primal-dual results, the optimal decision rules and their properties can be obtained and analyzed with relative ease.