Lecture 6 CONSTRAINED OPTIMIZATION

6.1. REVIEW [Roadmap_math1, comparative statics…]

6.2. CONSTRAINED OPTIMIZATION

6.2.1 MOTIVATION

Consider a firm producing one output $y \in \mathbf{R}$ using n inputs $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbf{R}^n$. Technological feasibility is defined by the set

$$\mathbf{F} = \{(y, \mathbf{x}): y \le g(\mathbf{x}): \mathbf{x} \ge 0, y \ge 0\},$$

where $g(\mathbf{x})$ is the production function, assumed to be twice continuously differentiable.

The firm purchases inputs $\mathbf{x}$ at market prices $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathbf{R}^n$, where $w_i > 0$ is the price of x_i. Then, $\mathbf{w} \cdot \mathbf{x} = \Sigma_i\, w_i \cdot x_i$ is cost of production for the firm. We want to consider the case where the firm wants to choose inputs $\mathbf{x}$ so as to minimize cost. Given some output $y > 0$, this is represented by the cost minimization problem

$$\underset{\mathbf{x}}{Min} \quad \{\mathbf{w} \cdot \mathbf{x} : (y, \mathbf{x}) \in \mathbf{F}\}$$

or

$$\underset{\mathbf{x}}{Min} \quad \{\mathbf{w} \cdot \mathbf{x} : y \le g(\mathbf{x}), \mathbf{x} \ge 0\}. \tag{1}$$

Denote the solution of this optimization problem by $\mathbf{x}^*$.

We will consider the (intuitive) situation where it is not possible to produce some positive output $y > 0$ without the use of some inputs. Under such a condition, given $y > 0$, the *optimal input use* $\mathbf{x}^*$ *must satisfy* $y = g(\mathbf{x}^*)$.

<u>Proof</u>: Assume that $y < g(\mathbf{x}^*)$. In general, by feasibility, $\mathbf{x}^* \ge 0$. Under the situation considered, producing a positive output y requires at least some positive inputs. Thus, $\mathbf{x}^*$ cannot be zero. Then, when $y < g(\mathbf{x}^*)$, it is always feasible to reduce the positive inputs in $\mathbf{x}^*$. But, with input prices being positive, this would reduce cost, contradicting that $\mathbf{x}^*$ is a cost-minimizing input levels. Thus, y cannot be less than $g(\mathbf{x}^*)$. Since technological feasibility implies that $y \le g(\mathbf{x}^*)$, it follows that $y = g(\mathbf{x}^*)$.

This means that the cost minimization problem (1) can be written equivalently as

$$\underset{\mathbf{x}}{Min} \quad \{\mathbf{w} \cdot \mathbf{x} : y = g(\mathbf{x}), \mathbf{x} \ge 0, \mathbf{x} \in \mathbf{R}^n\}.$$

This is a constrained minimization problem, with a single constraint: $y = g(\mathbf{x})$. We want to characterize its solution $\mathbf{x}^*$. Note that its solution $\mathbf{x}^*$ is the same as the solution of the constrained maximization problem

$$\underset{\mathbf{x}}{Max} \quad \{-\mathbf{w} \cdot \mathbf{x} : y = g(\mathbf{x}), \mathbf{x} \ge 0, \mathbf{x} \in \mathbf{R}^n\}.$$

In general, we want to analyze the solution of constrained optimization problems.

6.2.2 SOME MATHEMATICAL TOOLS

<u>Convex set</u>

Let $\mathbf{S}$ be a set defined to be a subset of $\mathbf{R}^n$: $\mathbf{S} \subset \mathbf{R}^n$.

The set $\mathbf{S}$ is *convex* if for any $\mathbf{x}^1$ and $\mathbf{x}^2 \in \mathbf{S}$, we have $[\theta \cdot \mathbf{x}^1 + (1-\theta) \cdot \mathbf{x}^2] \in \mathbf{S}$ for any $0 \le \theta \le 1$.

Quasi-Concavity

Let $f(\mathbf{x})$ be a function, where $\mathbf{x} \in \mathbf{R}^n$.

The function $f(\mathbf{x})$ is *quasi-concave* if, for any two points $\mathbf{x}^1$ and $\mathbf{x}^2$,

$$f(\mathbf{x}^2) \geq f(\mathbf{x}^1) \text{ implies that } f(\theta \cdot \mathbf{x}^1 + (1-\theta) \cdot \mathbf{x}^2) \geq f(\mathbf{x}^1),$$

for any θ, $0 \leq \theta \leq 1$.

The function $f(\mathbf{x})$ is *strictly quasi-concave* if, for any two points $\mathbf{x}^1$ and $\mathbf{x}^2$, $\mathbf{x}^1 \neq \mathbf{x}^2$,

$$f(\mathbf{x}^2) \geq f(\mathbf{x}^1) \text{ implies that } f(\theta \cdot \mathbf{x}^1 + (1-\theta) \cdot \mathbf{x}^2) > f(\mathbf{x}^1),$$

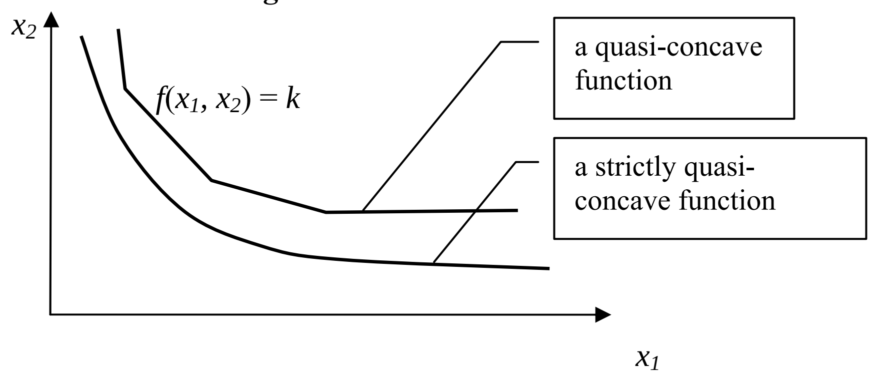
for any θ, $0 < \theta < 1$.

An *important result*:

The function $f(\mathbf{x})$ is *quasi-concave* if and only if $\{\mathbf{x}: f(\mathbf{x}) \geq k\}$ is a *convex set* for all k.

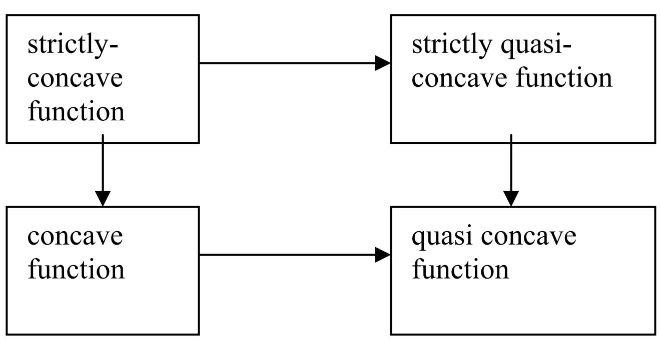
This is illustrated in figure 6.1 (where $f(\mathbf{x})$ is increasing in $\mathbf{x}$).

Figure 6.1



The relationships between quasi-concavity, strict quasi-concavity, concavity and strict concavity are presented in figure 6.2. It shows that *quasi-concavity is a weaker condition than concavity*: a (strictly) concave function is always (strictly) quasi-concave, but not vice versa.

Figure 6.2



A function $f(\mathbf{x})$ is *quasi-convex* if $-f(\mathbf{x})$ is quasi-concave.

And a function $f(\mathbf{x})$ is *strictly quasi-convex* if $-f(\mathbf{x})$ is strictly quasi-concave.

Some useful results:

- If $f(\mathbf{x})$ is quasi-concave and $F(f)$ is an increasing function, then $F(f(\mathbf{x}))$ is also quasi-concave.
- Assume that $f(\mathbf{x})$ is continuously differentiable.
  - The function $f(\mathbf{x})$ is quasi-concave *if and only if*
    $$f(\mathbf{x}^2) \geq f(\mathbf{x}^1) \text{ implies that } f'(\mathbf{x}^1) \cdot (\mathbf{x}^2 - \mathbf{x}^1) \geq 0$$
    for any two points $\mathbf{x}^1$ and $\mathbf{x}^2$. (Recall that $f_{\mathbf{x}}(\mathbf{x}^1) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ is a $(1\times n)$ vector, $\mathbf{x}$ is a $(n\times 1)$ vector, and $f_{\mathbf{x}}(\mathbf{x}^1)\cdot(\mathbf{x}^2 - \mathbf{x}^1) = \sum_i \frac{\partial f}{\partial x_i(\mathbf{x}^1)}\cdot(x_i^2 - x_i^1)$.

Let $f_i = \frac{\partial f}{\partial x_i}$, and $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. Define $\mathbf{C}_k(\mathbf{x}) = \begin{bmatrix} f_{11} & \cdots & f_{1k} & f_1 \\ \vdots & \ddots & \vdots & \vdots \\ f_{k1} & \cdots & f_{kk} & f_k \\ f_1 & \cdots & f_k & 0 \end{bmatrix}$ = $(k+1)\times(k+1)$ matrix, for some k, $1 \leq k \leq n$.

- Assume that $f(\mathbf{x})$ is twice continuously differentiable.
  - **If** the function $f(\mathbf{x})$ is quasi-concave, **then** $(-1)^k \cdot \det[\mathbf{C}_k(\mathbf{x})] \geq 0$ for $k = 1, \ldots, n$.
  - **If** $(-1)^k \cdot \det[\mathbf{C}_k(\mathbf{x})] > 0$ for all $\mathbf{x}$, for $k = 1, \ldots, n$, **then** $f(\mathbf{x})$ is quasi-concave.

<u>Note</u>: Similar results apply to quasi-convex functions (just replace $f(\mathbf{x})$ by $-f(\mathbf{x})$ in the above results…).

6.3 CONSTRAINED OPTIMIZATION

We consider the case of a constrained optimization problem, involving choosing a n-vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$, subject to m constraints. Throughout, we will assume that the number of constraints is less than the number of variables in $\mathbf{x}$, i.e., that $m < n$.

It will be convenient to start with the following *constrained maximization* problem

$$\underset{\mathbf{x}}{Max} \quad \{f(\mathbf{x}) : h_1(\mathbf{x}) = 0, \ldots h_m(\mathbf{x}) = 0; \mathbf{x} \geq 0; \mathbf{x} \in \mathbf{R}^n\},$$

where $f(\mathbf{x})$ is the *objective function*, and $h_j(\mathbf{x}) = 0$ is the j-th constraint, $j = 1, \ldots, m$, $m < n$. We will assume throughout that the functions $f(\mathbf{x})$ and $h_j(\mathbf{x})$, $j = 1, \ldots, m$, are twice continuous differentiable.

Let $\mathbf{h}(\mathbf{x}) = \begin{bmatrix} h_1(x) \\ \vdots \\ h_m(x) \end{bmatrix}$ denote a $(m\times 1)$ vector such that the m constraints can be written as $\mathbf{h}(\mathbf{x}) = 0$.

Then, the *constrained maximization* problem is

$$\underset{\mathbf{x}}{Max} \quad \{f(\mathbf{x}) : \mathbf{h}(\mathbf{x}) = 0; \mathbf{x} \geq 0; \mathbf{x} \in \mathbf{R}^n\}, \qquad (1)$$

Let $\mathbf{x}^*$ denote the solution to the constrained maximization problem. This means that $\mathbf{x}^*$ must be feasible and satisfy $\mathbf{h}(\mathbf{x}^*) = 0$, $\mathbf{x}^* \geq 0$. In addition, $\mathbf{x}^*$ is a global solution to the constrained maximization problem if and only if $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all feasible $\mathbf{x}$ satisfying $\mathbf{h}(\mathbf{x}) = 0$, $\mathbf{x} \geq 0$.

<u>Note</u>: If $\mathbf{x}^*$ is an *interior solution*, then $x_i^* > 0$, $i = 1, \ldots, n$. In this case, the constraint $\mathbf{x} \geq 0$ can be ignored without a loss of information. This will be convenient for the analysis presented

below. However, we should keep in mind that ignoring the constraint $\mathbf{x} \geq 0$ would not be appropriate if $\mathbf{x}^*$ is *not* an interior solution. Allowing for inequality restrictions in (1) will require a further generalization of the results presented here. This issue will be addressed later on in the class (in our discussion of the Kuhn-Tucker conditions).

Note: The constrained maximization problem can be easily converted to a constrained minimization problem by noting that $\underset{\mathbf{x}}{Max}\ \{f(\mathbf{x}) : \mathbf{h}(\mathbf{x}) = 0; \mathbf{x} \in \mathbf{R}^n\}$ as the same solution $\mathbf{x}^*$ as $\underset{\mathbf{x}}{Min}\ \{-f(\mathbf{x}) : \mathbf{h}(\mathbf{x}) = 0; \mathbf{x} \in \mathbf{R}^n\}$. As a result, all the results presented below can easily be applied to a minimization problem after replacing $f(\mathbf{x})$ by - $f(\mathbf{x})$.

6.4 Unconstrained formulation

In an attempt to solve the constrained optimization problem (1), it will be convenient to rely on the unconstrained optimization results we obtained earlier. This can be done by first solving the constraints $\mathbf{h}(\mathbf{x}) = 0$ and substituting the result into the objective function $f(\mathbf{x})$. The constraints $\mathbf{h}(\mathbf{x}) = 0$ constitute a system of m equations in n unknown $\mathbf{x}$. With $m < n$, this system of equations has an infinite number of solutions. To deal with this issue, *partition the vector* $\mathbf{x}$ *as* $\mathbf{x} = (\mathbf{x}_a, \mathbf{x}_b)$, where $\mathbf{x}_a = (x_1, \ldots, x_m) \in \mathbf{R}^m$ and $\mathbf{x}_b = (x_{m+1}, \ldots, x_n) \in \mathbf{R}^{n-m}$. Then, for each $\mathbf{x}_b$, consider solving the system of m equations $\mathbf{h}(\mathbf{x}_a, \mathbf{x}_b) = 0$ for the m variables $\mathbf{x}_a$. Denote this solution by

$$x_i = s_i(x_{m+1}, \ldots, x_n),\ i = 1, \ldots, m,$$

or equivalently,

$$\mathbf{x}_a = \mathbf{s}(\mathbf{x}_b),$$

where $\mathbf{x}_a = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$ is a $(m\times 1)$ vector and $\mathbf{s}(\mathbf{x}_b) = \begin{bmatrix} s_1(\mathbf{x}_b) \\ \vdots \\ s_m(\mathbf{x}_b) \end{bmatrix}$ is a $(m\times 1)$ vector.

Then, we can rely on the implicit function theorem. The *implicit function theorem* states that, if the $(m\times m)$ *Jacobian matrix* $\frac{\partial \mathbf{h}}{\partial \mathbf{x}_a} \equiv \mathbf{h}_a \equiv \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_m} \end{bmatrix}$ is *non-singular* (i.e., if $\det(\mathbf{h}_a) \neq 0$) when evaluated at $\mathbf{x} = (\mathbf{s}(\mathbf{x}_b), \mathbf{x}_b)$, then

1. there exists m *continuously differentiable* functions $\mathbf{s}(\mathbf{x}_b)$ such that $\mathbf{x}_a = \mathbf{s}(\mathbf{x}_b)$ is a solution to the system of equations $\mathbf{h}(\mathbf{x}_a, \mathbf{x}_b) = 0$, i.e. such that
$$\mathbf{h}(\mathbf{s}(\mathbf{x}_b), \mathbf{x}_b) = 0. \tag{2a}$$
2. the functions $\mathbf{s}(\mathbf{x}_b)$ have the following property
$$\frac{\partial \mathbf{s}}{\partial \mathbf{x}_b} \equiv \mathbf{s}_b = -[\mathbf{h}_a]^{-1} \cdot \mathbf{h}_b, \tag{2b}$$

where $\mathbf{s}_b \equiv \frac{\partial \mathbf{s}}{\partial \mathbf{x}_b} \equiv \begin{bmatrix} \frac{\partial s_1}{\partial x_{m+1}} & \cdots & \frac{\partial s_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_m}{\partial x_{m+1}} & \cdots & \frac{\partial s_m}{\partial x_n} \end{bmatrix}$ is a $m \times (n - m)$ matrix, $\mathbf{h}_a \equiv \frac{\partial \mathbf{h}}{\partial \mathbf{x}_a}$ is the $(m \times m)$ Jacobian matrix, and $\mathbf{h}_b \equiv \frac{\partial \mathbf{h}}{\partial \mathbf{x}_b} \equiv \begin{bmatrix} \frac{\partial h_1}{\partial x_{m+1}} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_{m+1}} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$ is a $m \times (n - m)$ matrix, all evaluated at $\mathbf{x} = (\mathbf{s}(\mathbf{x}_b), \mathbf{x}_b)$.

The non-singularity of the Jacobian matrix $\mathbf{h}_a$ (i.e., $\det(\mathbf{h}_a) \neq 0$) is crucial here: *it guarantees that we can always substitute the constraints into the objective function*. Indeed, if $\mathbf{h}_a$ were to be singular (with $\det(\mathbf{h}_a) = 0$), then we may not be able to solve the m constraints for $\mathbf{x}_a$ and the substitution procedure described below would breakdown.

Here, when evaluated at $\mathbf{x}^*$, the non-singularity of the Jacobian matrix $\mathbf{h}_a$ is given a special name: it is called the "constraint qualification."

Definition: *Constraint Qualification* (CQ):

The $(m \times m)$ matrix $\mathbf{h}_a(\mathbf{x}^*) \equiv \frac{\partial \mathbf{h}}{\partial \mathbf{x}_a(\mathbf{x}^*)} \equiv \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_m} \end{bmatrix}$ is *non-singular*: $\det[\mathbf{h}_a(\mathbf{x}^*)] \neq 0$.

Note: From linear algebra, $\det[\mathbf{h}_a(\mathbf{x}^*)] \neq 0$ implies that the $(m \times n)$ matrix

$\mathbf{h}_\mathbf{x}(\mathbf{x}^*) \equiv \frac{\partial \mathbf{h}}{\partial \mathbf{x}(\mathbf{x}^*)} \equiv \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$ must have *m linearly independent* columns. Note that, in the partition $\mathbf{x} = (\mathbf{x}_a, \mathbf{x}_b)$, the choice from $\mathbf{x}$ of the m variables that are included in $\mathbf{x}_a$ can be arbitrary. Then, given $n > m$, the constraint qualification CQ can be alternatively expressed as $\text{rank}[\mathbf{h}_\mathbf{x}(\mathbf{x}^*)] = m$. Indeed, one can always choose $\mathbf{x}_a$ such that it corresponds to m linearly independent columns of $\mathbf{h}_\mathbf{x}(\mathbf{x}^*)$. As a result, the CQ condition is sometimes expressed equivalently as the *rank condition*: $\text{rank}[\mathbf{h}_\mathbf{x}(\mathbf{x}^*)] = m$.

Then, if $\mathbf{x}^*$ is an interior solution and the constraint qualification CQ holds, we can always substitute $\mathbf{x}_a = \mathbf{s}(\mathbf{x}_b)$ into the objective function. This transforms the constrained optimization problem (1) into the ***equivalent*** *unconstrained optimization problem*

$$\underset{\mathbf{x}_b}{Max} \quad \{f(s(\mathbf{x}_b), \mathbf{x}_b) : \mathbf{x}_b \geq 0; \mathbf{x}_b \in \mathbf{R}^{n-m}\}. \tag{3}$$

We can now apply the tools available for unconstrained optimization to (3).

6.4.1 First order necessary condition (FONC)

If the constraint qualification (CQ) holds, the first order necessary condition (FONC) associated with (3) is

$$\frac{\partial f(\mathbf{s}(\mathbf{x}_b^*),\mathbf{x}_b^*)}{\partial \mathbf{x}_b} = 0.$$

Using the chain rule, this can be written as

$$\frac{\partial f}{\partial \mathbf{x}_a}\cdot\frac{\partial \mathbf{s}}{\partial \mathbf{x}_b} + \frac{\partial f}{\partial \mathbf{x}_b} = 0,$$

or

$$f_a(\mathbf{x}^*)\ \mathbf{s}_b(\mathbf{x}^*) + f_b(\mathbf{x}^*) = 0, \tag{4a}$$

where $f_a \equiv \frac{\partial f}{\partial \mathbf{x}_a} = (\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_m})$ is a $(1\times m)$ vector, $\mathbf{s}_b \equiv \frac{\partial \mathbf{s}}{\partial \mathbf{x}_b} \equiv \begin{bmatrix} \frac{\partial s_1}{\partial x_{m+1}} & \cdots & \frac{\partial s_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_m}{\partial x_{m+1}} & \cdots & \frac{\partial s_m}{\partial x_n} \end{bmatrix}$ is a $m\times(n - m)$ matrix,

and $f_b \equiv \frac{\partial f}{\partial \mathbf{x}_b} = (\frac{\partial f}{\partial x_{m+1}},\ldots,\frac{\partial f}{\partial x_n})$ is a $1\times(n - m)$ vector.

If the constraint qualification (CQ) holds, we know that FONC in (4a) is a *necessary* condition for $\mathbf{x}^* = (\mathbf{x}_a^*, \mathbf{x}_b^*) = (\mathbf{s}(\mathbf{x}_b^*), \mathbf{x}_b^*)$ to be a *interior solution to the maximization problem* (1) or (3).

The FONC (4a) constitutes a system of $(n - m)$ equations in $(n - m)$ unknowns $\mathbf{x}_b$. Thus, *if the constraint qualification (CQ) holds*, an interior solution to the maximization problem (1) or (3), $\mathbf{x}^* = (\mathbf{x}_a^*, \mathbf{x}_b^*)$, must be such that

- $\mathbf{x}_b^*$ satisfies the FONC (4a),
- $\mathbf{x}_a^*$ satisfies $\mathbf{x}_a^* = \mathbf{s}(\mathbf{x}_b^*)$. From (2a), this means that $\mathbf{x}^*$ must be feasible and satisfy the m constraints:

  $$\mathbf{h}(\mathbf{s}(\mathbf{x}_b^*), \mathbf{x}_b^*) = 0. \tag{4b}$$

  And from (2b),

  $$\frac{\partial \mathbf{s}}{\partial \mathbf{x}_b(\mathbf{x}_b^*)} \equiv \mathbf{s}_b(\mathbf{x}_b^*) = -[\mathbf{h}_a(\mathbf{x}^*)]^{-1}\cdot\mathbf{h}_b(\mathbf{x}^*). \tag{4c}$$

6.4.2 Second order necessary condition (SONC)

The second order necessary condition (SONC) associated with (3) is

$$\frac{\partial^2 f(\mathbf{s}(\mathbf{x}_b^*),\mathbf{x}_b^*)}{\partial \mathbf{x}_b^2} = \text{a } (n - m)\times(n - m) \text{ symmetric, \textit{negative semi-definite} matrix.} \tag{5}$$

If the constraint qualification (CQ) holds, we know that FONC (4a) and SONC (5) are *necessary* conditions for $\mathbf{x}^*$ to be an *interior solution to the maximization problem* (1) or (3).

6.4.3 Second order sufficient condition (SOSC)

If the constraint qualification (CQ) holds, the second-order sufficient condition (SOSC) is

$$\frac{\partial^2 f(\mathbf{s}(\mathbf{x}_b^*),\mathbf{x}_b^*)}{\partial \mathbf{x}_b^2} = \text{a } (n - m)\times(n - m) \text{ symmetric, \textit{negative definite} matrix,} \tag{6}$$

If the constraint qualification (CQ) holds, we know that FONC (4a) and SOSC (6) are *sufficient* conditions for $\mathbf{x}^*$ to be a *local interior solution to the maximization problem* (1) or (3).

6.4.4 Maximization under concavity

Assume that $\mathbf{x}_a = \mathbf{s}(\mathbf{x}_b)$ is a unique solution to the m constraints $\mathbf{h}(\mathbf{x}) = 0$, and that $f(\mathbf{s}(\mathbf{x}_b), \mathbf{x}_b)$ is a concave function of $\mathbf{x}_b$. Then, if the constraint qualification (CQ) holds, we know that the FONC

(4a) is a *necessary and sufficient* condition for **x*** to be an *interior solution to the maximization problem* (1) or (3).

6.5 The Lagrange approach

The unconstrained formulation presented above is nice. However, it requires solving systems of equations *in two sequential steps*: first solve the m constraints $\mathbf{h}(\mathbf{x}) = 0$ for $\mathbf{x}_a = \mathbf{s}(\mathbf{x}_b)$; and second solve the first order condition FONC in (4a) for $\mathbf{x}_b$*. This can be tedious. It would be nice to obtain the solution in *one step*. This is the method proposed by the mathematician Lagrange about two centuries ago.

Definition: Define the *Lagrange multipliers* to be the $(m\times 1)$ vector $\boldsymbol{\lambda}^*$, where

$$(\boldsymbol{\lambda}^*)^T = -f_a(\mathbf{x}^*)\cdot[\mathbf{h}_a(\mathbf{x}^*)]^{-1}, \tag{7}$$

where $f_a \equiv \frac{\partial f}{\partial \mathbf{x}_a}$ is a $1\times m$ matrix, and $\mathbf{h}_a \equiv \frac{\partial \mathbf{h}}{\partial \mathbf{x}_a}$ is a $m\times m$ matrix (which is invertible under the CQ condition).

Definition: Define the *Lagrangean* as the function

$$L(\mathbf{x}, \boldsymbol{\lambda}) \equiv f(\mathbf{x}) + \boldsymbol{\lambda}^T \mathbf{h}(\mathbf{x}), \tag{8}$$

$$\equiv f(\mathbf{x}) + \textstyle\sum_j \lambda_j h_j(\mathbf{x}),$$

where $\boldsymbol{\lambda}^T = (\lambda_1, \ldots, \lambda_m)$ is a $(1\times m)$ vector of linear weights for the m constraints $\mathbf{h}(\mathbf{x}) = 0$.

As we show next, our earlier results can now be expressed equivalently in terms of the Lagrangean function $L(\mathbf{x}, \boldsymbol{\lambda})$.

6.5.1 The first order necessary condition (FONC)

The FONC condition (4a) can be alternatively expressed as

$$\tfrac{\partial L}{\partial \mathbf{x}} \equiv L_{\mathbf{x}}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \equiv f_{\mathbf{x}}(\mathbf{x}^*) + (\boldsymbol{\lambda}^*)^T \cdot \mathbf{h}_{\mathbf{x}}(\mathbf{x}^*) = 0, \tag{9a}$$

$$\tfrac{\partial L}{\partial \lambda} \equiv L_{\lambda}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \equiv \mathbf{h}(\mathbf{x}^*)^T = 0, \tag{9b}$$

where $L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}^T \mathbf{h}(\mathbf{x})$.

Proof: Equation (9a) can be written in two parts: $f_a(\mathbf{x}^*) + (\boldsymbol{\lambda}^*)^T \mathbf{h}_a(\mathbf{x}^*) = 0$, and $f_b(\mathbf{x}^*) + (\boldsymbol{\lambda}^*)^T \mathbf{h}_b(\mathbf{x}^*) = 0$.

The first part of (9a) is equivalent to the definition of the Lagrange multipliers in (7) under the constraint qualification CQ.

The second part can be obtained as follows. Substituting (4c) into (4a) yields

$$f_b(\mathbf{x}^*) - f_a(\mathbf{x}^*)\,[\mathbf{h}_a(\mathbf{x}^*)]^{-1}\,\mathbf{h}_b(\mathbf{x}^*) = 0.$$

Using the definition of the Lagrange multipliers in (7), this gives $f_b(\mathbf{x}^*) + (\boldsymbol{\lambda}^*)^T \mathbf{h}_b(\mathbf{x}^*) = 0$, which is the second part of (9a).

Finally, equation (9b) guarantees that **x*** is feasible and satisfies the m constraints.

This shows that equations (9a) and (9b) are *equivalent* to the FONC condition (4a). This generates the following important result:

If the constraint qualification CQ holds, a necessary condition for **x*** *to be an interior solution to the maximization problem (1) is that (9a)-(9b) holds.*

As such, under CQ, (9a)-(9b) are the *first order necessary condition (FONC) for the Lagrangean formulation*. Note that equations (9a)-(9b) is a system of (n + m) equations in (n + m) unknowns: ($\mathbf{x}$, λ). In general we will be looking for a solution ($\mathbf{x}^*$, λ^*) of this system of equations. The reason is that, under CQ, an interior solution $\mathbf{x}^*$ to the constrained optimization problem must be a solution to the system of equations (9a)-(9b). In addition, solving (9a)-(9b) will also give us the value of the m Lagrange multipliers, λ^*. We will see later that obtaining the Lagrange multipliers λ^* as a by-product of solving (9a)-(9b) can provide useful information about the effects of the constraints $\mathbf{h}(\mathbf{x}) = 0$.

What is the difference between the two formulations?

- *Unconstrained approach*: Solving equation (4a) for $\mathbf{x}^*$ requires two steps: first solve the m constraints for $\mathbf{x}_a = \mathbf{s}(\mathbf{x}_b)$; second solve the FOC (4a) for $\mathbf{x}_b^*$.
- *Lagrange approach*: Solving (9a)-(9b) requires only one step: solve the system of (m + n) equations for ($\mathbf{x}^*$, λ^*). This comes at some cost: there are now (m + n) equations to solve. However, the Lagrange approach has one significant advantage: it gives the solution in one step.

Note: Both approaches require the constraint qualification CQ.

Note: For a *minimization* problem, $\underset{\mathbf{x}}{Min} \quad \{f(\mathbf{x}) : \mathbf{h}(\mathbf{x}) = 0; \mathbf{x} \geq 0; \mathbf{x} \in \mathbf{R}^n\}$, we obtain:

> *If the constraint qualification (CQ) holds, the FONC equations (4a)-(4b) are necessary conditions for* $\mathbf{x}^*$ *to be an interior solution to the minimization problem.*

6.5.2 Second order necessary condition (SONC)

The SONC (5) can be alternatively written as

$$[\mathbf{s}_b(\mathbf{x}_b^*)^T, \mathbf{I}_{n-m}] \cdot L_{\mathbf{xx}}(\mathbf{x}^*, \lambda^*) \cdot \begin{bmatrix} \mathbf{s}_b(\mathbf{x}_b^*) \\ \mathbf{I}_{n-m} \end{bmatrix} = \text{symmetric, } \textit{negative semi-definite} \text{ matrix,} \qquad (10)$$

where $\mathbf{s}_b = \frac{\partial \mathbf{s}}{\partial \mathbf{x}_b}$ is a $m \times (n - m)$ matrix, $\mathbf{I}_{n-m}$ is an identity matrix of dimension (n - m), and $L_{\mathbf{xx}} \equiv \frac{\partial^2 L}{\partial \mathbf{x}^2} \equiv f_{\mathbf{xx}} + \lambda^T \cdot \mathbf{h}_{\mathbf{xx}}$ is a ($n \times n$) matrix.

It follows that (10) is the *second order necessary condition (SONC) for the Lagrange formulation*. This generates the following important result:

> *If the constraint qualification CQ holds, necessary conditions for* $\mathbf{x}^*$ *to be an interior solution to the maximization problem (1) are that the FONC equations (9a)-(9b) and the SONC condition (10) hold.*

Proof: Denote the *feasible set* by $\mathbf{X} = \{\mathbf{x}: \mathbf{h}(\mathbf{x}) = 0, \mathbf{x} \geq 0, \mathbf{x} \in \mathbf{R}^n\}$. Thus, for any feasible $\mathbf{x} \in \mathbf{X}$, we have $\mathbf{h}(\mathbf{x}) = 0$, implying that $L(\mathbf{x}, \lambda) \equiv f(\mathbf{x}) + \lambda^T \mathbf{h}(\mathbf{x}) = f(\mathbf{x})$. Also, since $\mathbf{x}^* \in \mathbf{X}$, we have $L(\mathbf{x}^*, \lambda^*) = f(\mathbf{x}^*)$. When $\mathbf{x} \in \mathbf{X}$, taking a second order Taylor series expansion of $L(\mathbf{x}, \lambda)$ with respect to $\mathbf{x}$ in a *close neighborhood of* $\mathbf{x}^*$ gives

$$f(\mathbf{x}) = L(\mathbf{x}, \lambda) \cong f(\mathbf{x}^*) + L_{\mathbf{x}}(\mathbf{x}^*, \lambda^*) (\mathbf{x} - \mathbf{x}^*) + .5 (\mathbf{x} - \mathbf{x}^*)^T L_{\mathbf{xx}}(\mathbf{x}^*, \lambda^*) (\mathbf{x} - \mathbf{x}^*).$$

Under (9a)-(9b), $L_{\mathbf{x}}(\mathbf{x}^*, \lambda^*) = 0$, yielding

$$f(\mathbf{x}) - f(\mathbf{x}^*) \cong .5 (\mathbf{x} - \mathbf{x}^*)^T L_{\mathbf{xx}}(\mathbf{x}^*, \lambda^*) (\mathbf{x} - \mathbf{x}^*),$$

for $\mathbf{x} \in \mathbf{X}$ in a close neighborhood of $\mathbf{x}^*$. When $\mathbf{x}^*$ is an interior solution and the constraint qualification CQ holds, this means that $\mathbf{x} \equiv (\mathbf{x}_a, \mathbf{x}_b) = (\mathbf{s}(\mathbf{x}_b), \mathbf{x}_b)$, where $\mathbf{x}_b$ is in a close neighborhood of $\mathbf{x}_b^*$. Then, $\mathbf{x} - \mathbf{x}^* = \begin{bmatrix} \mathbf{s}(\mathbf{x}_b) - \mathbf{x}_a^* \\ \mathbf{x}_b - \mathbf{x}_b^* \end{bmatrix} \cong \begin{bmatrix} \mathbf{s}_b(\mathbf{x}_b^*)(\mathbf{x}_b - \mathbf{x}_b^*) \\ \mathbf{x}_b - \mathbf{x}_b^* \end{bmatrix} =$ $\begin{bmatrix} \mathbf{s}_b(\mathbf{x}_b^*) \\ \mathbf{I}_{n-m} \end{bmatrix} \cdot (\mathbf{x}_b - \mathbf{x}_b^*)$, where we used a first order Taylor series expansion of $\mathbf{s}(\mathbf{x}_b)$ evaluated at $\mathbf{x}_b^*$: $\mathbf{s}(\mathbf{x}_b) \cong \mathbf{x}_a^* + \mathbf{s}_b(\mathbf{x}^*)\,(\mathbf{x}_b - \mathbf{x}_b^*)$. This gives

$$f(\mathbf{x}) - f(\mathbf{x}^*) \cong .5(\mathbf{x}_b - \mathbf{x}_b^*)^T [\mathbf{s}_b(\mathbf{x}_b^*)^T, \mathbf{I}_{n-m}] L_{\mathbf{xx}}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \begin{bmatrix} \mathbf{s}_b(\mathbf{x}_b^*) \\ \mathbf{I}_{n-m} \end{bmatrix} (\mathbf{x}_b - \mathbf{x}_b^*).$$

Note that x* being a maximum means that $f(\mathbf{x}^*) \geq f(\mathbf{x})$, or $f(\mathbf{x}) - f(\mathbf{x}^*) \leq 0$, for all $\mathbf{x} \in \mathbf{X}$. Then, the above expression yields

$$(\mathbf{x}_b - \mathbf{x}_b^*)^T [\mathbf{s}_b(\mathbf{x}_b^*)^T, \mathbf{I}_{n-m}] L_{\mathbf{xx}}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \begin{bmatrix} \mathbf{s}_b(\mathbf{x}_b^*) \\ \mathbf{I}_{n-m} \end{bmatrix} (\mathbf{x}_b - \mathbf{x}_b^*) \leq 0,$$

for $\mathbf{x}_b$ in a close neighborhood of $\mathbf{x}_b^*$. When $\mathbf{x}^*$ is an interior solution, except for being "small," the choice of $(\mathbf{x}_b - \mathbf{x}_b^*)$ is arbitrary, implying the negative semi-definiteness in (10). (The symmetry property in (10) follows from Young theorem). Thus, under CQ, an interior maximum x* implies (10).

Note: For a *minimization* problem, $\underset{\mathbf{x}}{Min}\ \{f(\mathbf{x}) : \mathbf{h}(\mathbf{x}) = 0; \mathbf{x} \geq 0; \mathbf{x} \in \mathbf{R}^n\}$, we obtain the following result. The *second order necessary conditions* (SONCmin) are

$$[\mathbf{s}_b(\mathbf{x}_b^*)^T, \mathbf{I}_{n-m}] L_{\mathbf{xx}}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \begin{bmatrix} \mathbf{s}_b(\mathbf{x}_b^*) \\ \mathbf{I}_{n-m} \end{bmatrix} = \text{symmetric, } \textit{positive semi-definite} \text{ matrix.} \qquad (10')$$

Then, *if the constraint qualification CQ holds, a necessary condition for* $\mathbf{x}^*$ *to be an interior solution to the minimization problem is that the FONC equations (9a)-(9b) and the SONCmin condition (10') hold.*

6.6 Second order sufficient condition (SOSC)

The SOSC in (6) can be alternatively written as

$$[\mathbf{s}_b(\mathbf{x}_b^*)^T, \mathbf{I}_{n-m}] L_{\mathbf{xx}}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \begin{bmatrix} \mathbf{s}_b(\mathbf{x}_b^*) \\ \mathbf{I}_{n-m} \end{bmatrix} = \text{symmetric, } \textit{negative definite} \text{ matrix.} \qquad (11)$$

It follows that (11) is the *second order sufficient condition (SOSC) for the Lagrange formulation*. This generates the following important result:

If the constraint qualification CQ holds, a sufficient condition for x to be a local interior solution to the maximization problem (1) is that the FONC equations (9a)-(9b) and the SOSC condition (11) hold.*

Proof: Follow the same steps as in the previous proof. Given CQ, an interior solution, and under (9a)-(9b), we have

$$f(\mathbf{x}) - f(\mathbf{x}^*) \cong .5(\mathbf{x}_b - \mathbf{x}_b^*)^T [\mathbf{s}_b(\mathbf{x}_b^*)^T, \mathbf{I}_{n-m}] L_{\mathbf{xx}}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \begin{bmatrix} \mathbf{s}_b(\mathbf{x}_b^*) \\ \mathbf{I}_{n-m} \end{bmatrix} (\mathbf{x}_b - \mathbf{x}_b^*),$$

where $\mathbf{x}_b$ is in a close neighborhood of $\mathbf{x}_b$*. Under (11), the quadratic form on the right-hand side is negative. Thus, under (11), the left-hand side must also be negative, implying that $f(\mathbf{x}^*) > f(\mathbf{x})$ for any feasible **x** in the close neighborhood of **x***. This implies that **x*** is a local solution to the constrained maximization problem.

Note: For a *minimization* problem, $\underset{\mathbf{x}}{Min} \ \{f(\mathbf{x}) : \mathbf{h}(\mathbf{x}) = 0; \mathbf{x} \geq 0; \mathbf{x} \in \mathbf{R}^n\}$, the *second order sufficient conditions* (SOSCmin) are

$$[\mathbf{s}_b(\mathbf{x}_b^*)^T, \mathbf{I}_{n-m}] L_{\mathbf{xx}}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \begin{bmatrix} \mathbf{s}_b(\mathbf{x}_b^*) \\ \mathbf{I}_{n-m} \end{bmatrix} = \text{symmetric, \textit{positive definite} matrix.} \quad (11')$$

Then, *if the constraint qualification CQ holds, a sufficient condition for* **x*** *to be a local interior solution to the minimization problem is that the FONC equations (9a)-(9b) and the SOSCmin condition (11') hold.*

6.7 Optimization under quasi-concavity

Consider the case where the functions $f(\mathbf{x})$, and $h_j(\mathbf{x})$, $j = 1, \ldots, m$, are *quasi-concave*. Then, the following result applies.

If f and h are quasi-concave functions, λ ≥ 0, and either f is concave or $f_{\mathbf{x}}(\mathbf{x}^*) \neq 0$, then the FONC (9a)-(9b) are sufficient to identify* **x*** *as a global interior solution to the maximization problem (1).*

Proof: Under the quasi-concavity of the function $h_j(\mathbf{x})$, we have seen that

$h_j(\mathbf{x}) \geq h_j(\mathbf{x}^*)$ implies that $\frac{\partial h_i}{\partial \mathbf{x}(\mathbf{x}^*)}[\mathbf{x} - \mathbf{x}^*] \geq 0$, $j = 1, \ldots, m$.

Denote the feasible set by $\mathbf{X} \equiv \{\mathbf{x}: \mathbf{h}(\mathbf{x}) = 0, \mathbf{x} \geq 0, \mathbf{x} \in R^n\}$. Note that $h_j(\mathbf{x}^*) = 0$ and $h_j(\mathbf{x}) = 0$ for all feasible $\mathbf{x} \in \mathbf{X}$. It follows that, under the quasi-concavity of the functions $\mathbf{h}(\mathbf{x})$,

$\mathbf{h}_{\mathbf{x}}(\mathbf{x}^*)\,[\mathbf{x} - \mathbf{x}^*] \geq 0$, for all feasible $\mathbf{x} \in \mathbf{X}$. (12)

When $\boldsymbol{\lambda}^* \geq 0$, (12) implies

$\boldsymbol{\lambda}^{*T}\, \mathbf{h}_{\mathbf{x}}(\mathbf{x}^*)\,[\mathbf{x} - \mathbf{x}^*] \geq 0$, for all feasible $\mathbf{x} \in \mathbf{X}$. (13)

Under the FONC (9a)-(9b),

$L_{\mathbf{x}}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \equiv f_{\mathbf{x}}(\mathbf{x}^*) + \boldsymbol{\lambda}^{*T}\, \mathbf{h}_{\mathbf{x}}(\mathbf{x}^*) = 0$,

or

$f_{\mathbf{x}}(\mathbf{x}^*) = -\boldsymbol{\lambda}^{*T}\, \mathbf{h}_{\mathbf{x}}(\mathbf{x}^*)$,

implying that

$f_{\mathbf{x}}(\mathbf{x}^*)\,[\mathbf{x} - \mathbf{x}^*] = -\boldsymbol{\lambda}^{*T}\, \mathbf{h}_{\mathbf{x}}(\mathbf{x}^*)\,[\mathbf{x} - \mathbf{x}^*]$. (14)

Combining (13) and (14) yields

$f_{\mathbf{x}}(\mathbf{x}^*)\,[\mathbf{x} - \mathbf{x}^*] \leq 0$, for all feasible $\mathbf{x} \in \mathbf{X}$. (15)

Case 1: $f(\mathbf{x})$ is concave. We have seen that the concavity of $f(\mathbf{x})$ implies that $f_{\mathbf{x}}(\mathbf{x}^*)\,[\mathbf{x} - \mathbf{x}^*] \geq f(\mathbf{x}) - f(\mathbf{x}^*)$. Combining this with (15) gives $f(\mathbf{x}) - f(\mathbf{x}^*) \leq 0$, or $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all feasible $\mathbf{x} \in \mathbf{X}$. This implies that **x*** is a global maximum.

Case 2: $f_\mathbf{x}(\mathbf{x}^*) \neq 0$. This implies that $\frac{\partial f}{\partial x_i(\mathbf{x}^*)} \neq 0$ for some i, $i = 1, \ldots, n$. Choose any feasible $\mathbf{x} \in \mathbf{X}$ satisfying $x_i \neq x_i^*$ (where i corresponds to $\frac{\partial f}{\partial x_i(\mathbf{x}^*)} \neq 0$). It follows from (15) that

$$f_\mathbf{x}(\mathbf{x}^*) [\mathbf{x} - \mathbf{x}^*] < 0. \quad (16)$$

We have seen that, under the quasi-concavity of $f(\mathbf{x})$: $f(\mathbf{x}) \geq f(\mathbf{x}^*)$ implies that $f_\mathbf{x}(\mathbf{x}^*) [\mathbf{x} - \mathbf{x}^*] \geq 0$. Alternatively stated, under the quasi-concavity of $f(\mathbf{x})$: $f_\mathbf{x}(\mathbf{x}^*) [\mathbf{x} - \mathbf{x}^*] < 0$ implies that $f(\mathbf{x}) < f(\mathbf{x}^*)$. Thus, from (16), the quasi-concavity of $f(\mathbf{x})$ implies that $f(\mathbf{x}) < f(\mathbf{x}^*)$ for any feasible $\mathbf{x} \in \mathbf{X}$ satisfying $x_i \neq x_i^*$. Note that this result still applies when $x_i \neq x_i^*$ is chosen "arbitrarily close" to x_i^*. Under the continuity of $f(\mathbf{x})$, this means that the quasi-concavity of $f(\mathbf{x})$ implies that that $f(\mathbf{x}) \leq f(\mathbf{x}^*)$ for all feasible $\mathbf{x} \in \mathbf{X}$, i.e. that $\mathbf{x}^*$ is a global maximum.

This states that, *when the functions $f(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$ are quasi-concave, then the FONC conditions (9a)-(9b) are sufficient to identify $\mathbf{x}^*$ as a global interior maximum* ***if $\lambda^* \geq 0$, and either f is concave or $f_\mathbf{x}(\mathbf{x}^*) \neq 0$***.

Recall our earlier result:

> ***If the constraint qualification CQ holds****, then FONC (9a)-(9b) are necessary for $\mathbf{x}^*$ to be a global interior solution to the maximization problem (1).*

This means that, under quasi-concavity of $f(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$, the FONC (9a)-(9b) are in general *neither necessary nor sufficient* to identify an interior solution to the maximization problem (1). However, the FONC become *necessary* conditions when the constraint qualification CQ holds. And, under quasi-concavity of $f(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$, the FONC become a *sufficient* condition when $\lambda^* \geq 0$, and either f is concave or $f_\mathbf{x}(\mathbf{x}^*) \neq 0$. This indicates that the quasi-concavity of $f(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$ is "*almost* all we need" (but *not quite*!) to rely on the FONC conditions (9a)-(9b) when we want to identify a global solution to the constrained maximization problem (1).

6.8 Summary

These results suggests the following strategy to identify an interior solution to the *constrained maximization* problem (1), $\underset{\mathbf{x}}{Max} \; \{f(\mathbf{x}) : \mathbf{h}(\mathbf{x}) = 0; \mathbf{x} \geq 0; \mathbf{x} \in \mathbf{R}^n\}$,

- Investigate if the constraint qualification CQ is satisfied. (If CQ does not hold, the procedure below can break down).
- If CQ is satisfied, find $\mathbf{x}^*$ satisfying the FONC (9a)-(9b) in the Lagrange method. (This involves solving a system of $n + m$ equations in $n + m$ unknown, $(\mathbf{x}^*, \lambda^*)$).
- If the SONC (10) are not satisfied at $(\mathbf{x}^*, \lambda^*)$, then $\mathbf{x}^*$ cannot be an interior solution to the maximization problem.
- If CQ holds and the SOSC (11) are satisfied at $(\mathbf{x}^*, \lambda^*)$, then $\mathbf{x}^*$ is a *local interior solution* to the maximization problem (1). (Note that this does not imply that it is global solution…)
- If CQ holds, $f(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$ are *quasi-concave* functions, $\lambda^* \geq 0$, and either $f(\mathbf{x})$ is concave or $f_\mathbf{x}(\mathbf{x}^*) \neq 0$, then $\mathbf{x}^*$ is a *global interior solution* to the maximization problem (1).

Checking the SOSC (11):

- either find (using a computer) the characteristic values (or eigenvalues) of the matrix in (11) and check that they are all *negative*;
- or check the following determinants. Define the *bordered Hessian* as the $(n + m)\times(n + m)$ matrix $\mathbf{H} \equiv \frac{\partial^2 L}{\partial(\mathbf{x},\boldsymbol{\lambda})^2} \equiv \begin{bmatrix} L_{\mathbf{xx}} & L_{\mathbf{x\lambda}} \\ L_{\boldsymbol{\lambda}\mathbf{x}} & L_{\boldsymbol{\lambda\lambda}} \end{bmatrix} \equiv \begin{bmatrix} L_{\mathbf{xx}} & \mathbf{h}_{\mathbf{x}}^T \\ \mathbf{h}_{\mathbf{x}} & \mathbf{0}_m \end{bmatrix}$, evaluated at $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$, where $L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}^T \mathbf{h}(\mathbf{x})$. Define the $(m + k)\times(m + k)$ matrix $\mathbf{C}_k$ from $\mathbf{H}$ by deleting from $\mathbf{H}$ the *first (n-k) rows and (n-k) columns*, $k = m+1, \ldots, n$. Then,

  check that $(-1)^k \det(\mathbf{C}_k) > 0$, for $k = m+1, \ldots, n$.

  Note that, with $\mathbf{C}_n = \mathbf{H}$, this always implies $\det(\mathbf{H}) \neq 0$.
  When $m = 0$ (the *unconstrained case*), this gives $\det(\mathbf{C}_k) < 0$ for $k = 1$, > 0 for $k = 2$, < 0 for $k = 3$, etc.

  In the single variable case ($n = 1$), this gives: $f_{\mathbf{xx}}(\mathbf{x}^*) < 0$.

  In the two-variable case ($n = 2$), this gives: $f_{22} < 0$ for $k = 1$, and $\det\begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix} = f_{11} f_{22} - f_{12}^2 > 0$ for $k = 2$, where $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ evaluated at $\mathbf{x}^*$. (Note that $f_{22} < 0$ and $f_{22} f_{11} > f_{12}^2 \geq 0$ imply that $f_{11} < 0$).

  When $m = 1$ (*single constraint*), this gives $\det(\mathbf{C}_k) > 0$ for $k = 2$, < 0 for $k = 3$, > 0 for $k = 4$, etc.

  In the two-variable case ($n = 2$), this gives: $\det(\mathbf{H}) = \det\begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial h_1}{\partial x_1} \\ \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & 0 \end{bmatrix} > 0$, all expressions being evaluated at $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$. (Note that this does *not* imply that $\frac{\partial^2 L}{\partial x_i^2} < 0$, $i = 1, 2$).

  And so on…

Similarly, you can use the following strategy to identify an interior solution to the *constrained minimization* problem, $\underset{\mathbf{x}}{Min} \ \{ f(\mathbf{x}) : \mathbf{h}(\mathbf{x}) = 0; \mathbf{x} \geq 0; \mathbf{x} \in \mathbf{R}^n \}$,

- Investigate if the constraint qualification CQ is satisfied. (If CQ does not hold, the procedure below can break down).
- If CQ holds, find $\mathbf{x}^*$ satisfying the FONC (9a)-(9b) in the Lagrange method. (This involves solving a system of $n + m$ equations for $n + m$ unknown $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$.
- If SONCmin (10') are not satisfied at $\mathbf{x}^*$, then $\mathbf{x}^*$ cannot be an interior solution to the minimization problem.
- If CQ holds and SOSCmin (11') are satisfied at $\mathbf{x}^*$, then $\mathbf{x}^*$ is a *local interior solution* to the minimization problem. (Note that this does not imply that it is global solution…)
- If CQ holds, $f(\mathbf{x})$ and $\mathbf{h}(\mathbf{x})$ are *quasi-convex* functions, $\boldsymbol{\lambda}^* \geq 0$, and either $f(\mathbf{x})$ is convex or $f_{\mathbf{x}}(\mathbf{x}^*) \neq 0$, then $\mathbf{x}^*$ is a *global interior solution* to the minimization problem.

Checking the SOSCmin (11'):

- either find (using a computer) the eigenvalues of the matrix in (11') and check that they are all *positive*;
- or, using the matrices $\mathbf{C}_k$ defined above,

  check that $(-1)^m \det(\mathbf{C}_k) > 0$, *for* $k = m+1, \ldots, n$.

Note that, with $\mathbf{C}_n = \mathbf{H}$, this always implies $\det(\mathbf{H}) \neq 0$.

When $m = 0$ (the *unconstrained* case), this gives $\det(\mathbf{C}_k) > 0$, $k = 1, \ldots, n$.

In the single variable case ($n = 1$), this gives: $f_{\mathbf{xx}}(\mathbf{x}^*) > 0$.

In the two-variable case ($n = 2$), this gives: $f_{22} > 0$ for $k = 1$, and $\det\begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix} =$ $f_{11} f_{22} - f_{12}^2 > 0$ for $k = 2$, where $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ evaluated at $\mathbf{x}^*$. (Note that $f_{22} > 0$ and $f_{22} f_{11} > f_{12}^2 \geq 0$ imply that $f_{11} > 0$).

When $m = 1$ (*single constraint*), this gives $\det(\mathbf{C}_k) < 0$, $k = 2, \ldots, n$.

In the two-variable case ($n = 2$), this gives: $\det(\mathbf{H}) = \det\begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial h_1}{\partial x_1} \\ \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & 0 \end{bmatrix} < 0$, all expressions being evaluated at $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$. (Note that this does *not* imply that $\frac{\partial^2 L}{\partial x_i^2} > 0$, $i = 1, 2$).

And so on…