

## Lecture 5 SOME APPLICATIONS

## 5.1 REVIEW

[Profit maximization of  $n$ -inputs, 2-inputs, general case...]

## 5.2 SOME APPLICATIONS

## A. Relationship between a marginal and an average function

Let  $f(x)$  be some differentiable function, where  $x \in \mathbb{R}$ ,  $x > 0$ . Then, the *average* is defined as  $A(x) = \frac{f(x)}{x}$ , and the *marginal* is defined as  $M(x) = f'(x)$ . How are these related?

By definition,  $f(x) = A(x) \cdot x$ .

Differentiate (using the product rule) to obtain

$$M(x) = f'(x) = A'(x) \cdot x + A(x)$$

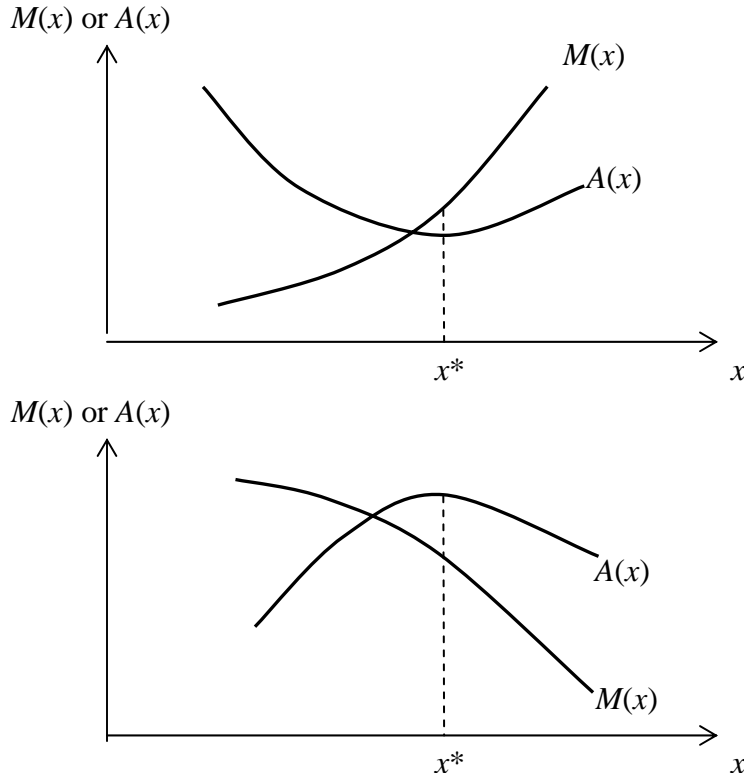
or, for  $x > 0$ ,

$$A'(x) = \frac{1}{x} \cdot (M(x) - A(x)).$$

- If the average  $A(x)$  has an interior minimum or maximum  $x^* > 0$ , then at this point  $x^*$ ,  $A'(x^*) = 0$ , and  $M(x) = A(x)$ ,
- At points where  $A(x)$  is decreasing in  $x$  (with  $A'(x) < 0$ ), then  $M(x) < A(x)$ ,
- At points where  $A(x)$  is increasing in  $x$  (with  $A'(x) > 0$ ), then  $M(x) > A(x)$ .

These results apply to all differentiable functions  $f(x)$ .

Figure 5.1



(Special case: what if  $f(x) = w \cdot x$ ? Then,  $A(x) = w = M(x)$  and  $A'(x) = 0$  everywhere).

### B. The monopolist's problem

A monopolist is a firm facing a downward sloping demand curve such that any increase (decrease) in its production tends to lower (raise) output price. Denote by  $p(y)$  the price dependent demand function facing the monopolist, where  $y$  is the firm output and  $p'(y) < 0$ . Then, its revenue is  $R(y) = p(y) \cdot y$ . Let its cost be  $C(y)$ . Assume that the cost function is increasing:  $C'(y) > 0$ . The monopolist profit is:  $\pi(y) = R(y) - C(y) = p(y) \cdot y - C(y)$  for some output  $y \geq 0$ .

The profit-maximizing monopolist would choose  $y$  as follows

$$\text{Max}_y \{p(y) y - C(y): y \geq 0\}.$$

Under differentiability, the necessary condition for an interior solution  $y^* > 0$  to this maximization problem is

$$\text{FONC: } \pi'(y^*) = R'(y^*) - C'(y^*) = p(y^*) + p'(y^*) y^* - C'(y^*) = 0,$$

or

$$\pi'(y^*) = R'(y^*) - C'(y^*) = 0,$$

or

$$p(y^*) + p'(y^*) y^* = C'(y^*).$$

The left-hand side is the marginal revenue  $MR = R'(y^*) = p(y^*) + p'(y^*) y^*$ . The right-hand side is the marginal cost  $MC = C'(y^*)$ . Thus, the FONC indicate that, at the profit maximizing solution  $y^*$ ,  $MR = MC$  where the monopoly price is  $p^m = p(y^*)$ .

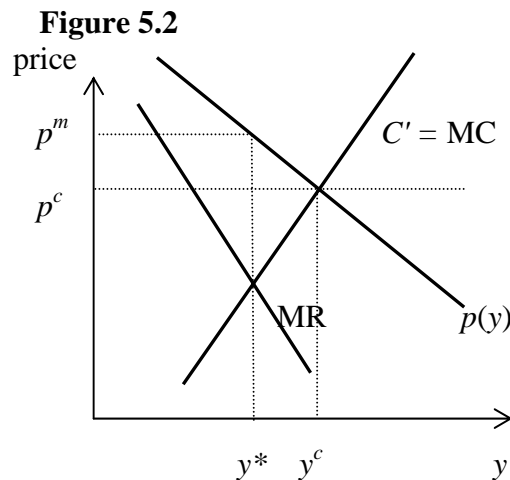


Figure 5.2 illustrates that a monopolist supplies less and charges more than a competitive firm. To see that, note that FONC implies that  $p(y^*) > C'(y^*)$  when  $p'(y^*) < 0$ . But  $p^c = C'(y^c)$  under competition (we will see this result in our analysis of cost minimizing behavior later). Then,  $C' > 0$  implies that  $p(y^*) > p^c$  and  $y^* < y^c$ .

Finally, note that FONC does not guarantee profit maximization. For this, we need, SONC/SOSC:

$$\text{SONC: } \pi''(y^*) = \frac{\partial MR}{\partial y} - \frac{\partial MC}{\partial y} \leq 0.$$

This means that MR must cross MC from above, as the slope of the MR function must be less than the slope of the MC function.

### C. Dynamic Choice

#### C1. Discounting

Suppose you invest \$1 at an annual interest rate  $r > 0$ . At the end of the year, you will have  $\$y = (1+r)$ . After two years, you will have  $(1+r)^2$ . And after  $t$  years, you will have  $(1+r)^t$ . It means that receiving \$1  $t$  years from now is worth at the present time:

$\frac{1}{(1+r)^t} = (1+r)^{-t}$ . The term  $\frac{1}{(1+r)^t}$ , is called a *discount factor* in *discrete time*. And the *present value* of receiving  $\$ \pi$   $t$  years from now is:  $\frac{\pi}{(1+r)^t}$ .

Now suppose that the bank pays interest at rate  $r$  compounded semi-annually. Then every 6 months it pays  $\frac{r}{2}$  percent on the amount in the account.

After six months, you receive:  $1 + \frac{r}{2}$ .

After 1 year, you receive:  $y = (1 + \frac{r}{2}) \cdot (1 + \frac{r}{2}) = (1 + \frac{r}{2})^2$ .

Next suppose that the bank pays interest quarterly. Then, after a year, you receive  $y = (1 + \frac{r}{4})^4$  ... In general, if the bank pays  $n$  times per year, at the end of the year, you receive  $y = (1 + \frac{r}{n})^n$ . As  $n$  becomes large, this is called *continuous compounding*. Taking the limit of this expression as  $n \rightarrow \infty$ , we obtain  $y = \lim_{n \rightarrow \infty} (1 + \frac{r}{n})^n = e^r$ , where  $e^r$  is the *exponential function* of  $r$ .

It follows that, under *continuous compounding*, \$1 invested now at an annual interest rate  $r > 0$  is worth  $\$e^r$  after a year, and  $\$e^{rt}$  after  $t$  years. Alternatively, \$1 received  $t$  years from now is worth at the present time:  $\frac{1}{e^{rt}} = e^{-rt}$ . Thus,  $e^{-rt}$  is the *discount factor* in *continuous time*. And the *present value* of receiving  $\$ \pi$   $t$  years from now is:  $\pi \cdot e^{-rt}$ .

Some important properties of the exponential function  $e^x$  are:

- if  $f(x) = e^x$ , then  $f'(x) = e^x$ .
- if  $f(x) = e^{kx}$ , then  $f'(x) = k \cdot e^{kx}$ .

Finally, given  $|a| < 1$ , note that the infinite sum  $S = 1 + a + a^2 + a^3 + \dots = \frac{1}{1-a}$ .

#### C2. Optimal timing

Let  $t = 0$  denote current time. Suppose that an asset grows in value over time, its *market value* at time  $t$  being  $v(t)$ . Suppose that  $v'(t) > 0$  and  $v''(t) < 0$ , so that the asset value grows at a decreasing rate. Assume that the only value of the asset is its market value  $v(t)$ . What is the optimal time to sell the asset to maximize its present value?

Let  $r$  be the interest rate. The *present value* of selling the asset at time  $t$  and receiving income  $v(t)$  at time  $t$  is  $P(t) = v(t) \cdot e^{-rt}$ , where  $r > 0$  is the interest rate.

We want to know the optimum selling time  $t^*$  that maximizes the present value  $P(t)$ . Assuming an interior solution, find the value  $t^* > 0$  satisfying the FONC:

$$P'(t) = v'(t) \cdot e^{-rt} - r \cdot v(t) \cdot e^{-rt} = 0,$$

or, since  $e^{-rt} > 0$ ,

$$v'(t) = r \cdot v(t),$$

or

$$r = \frac{v'(t)}{v(t)}.$$

This implies that, at the optimum selling time  $t^* > 0$ , the interest rate  $r$  must be equal to the growth rate of return  $\frac{v'(t)}{v(t)}$ . Here the interest rate  $r$  measures the opportunity cost of money. Then FONC states that, at the margin, *the growth rate of return must be equal to the interest rate*. Note that this applies only to assets that are *not replaced*.

### C3. Optimal replacement

Now consider the case where an asset is purchased at time  $t = 0$  at a cost  $C \geq 0$ . Again, the asset grows in value over time, its *market value* at time  $t$  being  $v(t)$ , with  $v'(t) > 0$  and  $v''(t) < 0$ . Assume that the only value of the asset is its market value  $v(t)$ . Then, the present value of the current asset sold at time  $t$  is

$$\pi(t) = v(t) e^{-rt} - C,$$

where  $r > 0$  is the interest rate. Also, assume that a new asset is purchased (at a cost  $C$ ) right after the previous asset is sold at time  $t$  for a value  $v(t)$ . In this context,  $t$  denotes the replacement time of the asset.

Example: The case of *forest management* where trees are the asset and their only value is their timber value. The trees are planted (or replanted) on a fixed amount of land every  $t$  periods;  $C$  is the planting cost; and  $v(t)$  is the market value of the trees when they are harvested at time  $t$ . In this context,  $t$  is the harvest time for trees.

The first asset yields a present value at time  $t = 0$  of  $\pi(t)$ . The second asset (replacing the first one) has a present value  $\pi(t) \cdot e^{-rt}$ . The third asset (replacing the second one) gives a present value  $\pi(t) \cdot e^{-2rt}$ , etc.... Assuming an *infinite planning horizon*, it follows that the present value of an infinite succession of assets is

$$P(t) = \pi(t) + \pi(t) \cdot e^{-rt} + \pi(t) \cdot e^{-2rt} + \dots = \pi(t)(1 + e^{-rt} + e^{-2rt} + \dots) = \frac{\pi(t)}{1 - e^{-rt}},$$

given that  $|e^{-rt}| < 1$ , as derived above.

We would like to know what is the optimal replacement time  $t^*$  that maximizes the present value  $P(t)$ . Assuming an interior solution  $t^* > 0$ , the FONC associated with the maximization of  $P(t)$  is

$$P'(t) = 0,$$

or

$$\frac{\pi'(t)}{1 - e^{-rt}} - \frac{r \cdot e^{-rt} \cdot \pi(t)}{(1 - e^{-rt})^2} = 0,$$

or

$$\pi'(t) \cdot (1 - e^{-rt}) = r \cdot e^{-rt} \cdot \pi(t),$$

or, given  $\pi(t) = v(t) \cdot e^{-rt} - C$ ,

$$(v'(t) \cdot e^{-rt} - r \cdot e^{-rt} \cdot v(t)) \cdot (1 - e^{-rt}) = r \cdot e^{-rt} \cdot (v(t) \cdot e^{-rt} - C),$$

or, after dividing by  $e^{-rt}$  and simplifying,

$$v'(t) \cdot (1 - e^{-rt}) = r \cdot (v(t) - C),$$

or, assuming  $v(t) - C > 0$ ,

$$\frac{v'(t)}{v(t) - C} = \frac{r}{1 - e^{-rt}}.$$

The FONC states that, at the optimal replacement time  $t^* > 0$ , the rate of change in asset value,  $\frac{v'(t)}{v(t) - C}$ , must equal the opportunity cost of replacement,  $\frac{r}{1 - e^{-rt}}$ . This is called the *Faustmann criterion*. Note that  $\frac{r}{1 - e^{-rt}} > r$ , reflecting that the opportunity cost is higher than the interest rate (because it also includes the opportunity cost of not replanting).