A. The monopolist's problem in the output market

A monopolist is a firm facing a downward sloping demand curve such that any increase (decrease) in its production tends to lower (raise) output price. Denote by $p(y)$ the price dependent demand function facing the monopolist, where y is the firm output and $p'(y) < 0$. Then, its revenue is $R(y) = p(y) \cdot y$. Let its cost be $C(y)$. Assume that the cost function is increasing: $C'(y) > 0$. The monopolist profit is: $\pi(y) = R(y) - C(y) = p(y) \cdot y - C(y)$ for some output $y \geq 0$.

The profit-maximizing monopolist would choose y as follows

$$\text{Max}_y \{p(y)\, y - C(y)\text{: } y \geq 0\}.$$

Under differentiability, the necessary condition for an interior solution $y^* > 0$ to this maximization problem is

FONC: $\pi'(y^*) = R'(y^*) - C'(y^*) = p(y^*) + p'(y^*)\, y^* - C'(y^*) = 0,$

or

$$\pi'(y^*) = R'(y^*) - C'(y^*) = 0,$$

or

$$p(y^*) + p'(y^*)\, y^* = C'(y^*).$$

The left-hand side is the marginal revenue MR = $R'(y^*) = p(y^*) + p'(y^*)\, y^*$. The right-hand side is the marginal cost MC = $C'(y^*)$. Thus, the FONC indicate that, at the profit maximizing solution y^*, MR = MC where the monopoly price is $p^m = p(y^*)$.

Figure A1

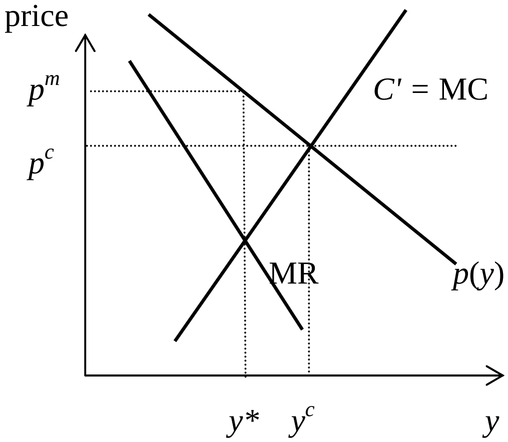

Figure A1 illustrates that a monopolist supplies less and charges more than a competitive firm. To see that, note that FONC implies that $p(y^*) > C'(y^*)$ when $p'(y^*) < 0$. But $p^c = C'(y^c)$ under competition (we will see this result in our analysis of cost minimizing behavior later). Then, $C' > 0$ implies that $p(y^*) > p^c$ and $y^* < y^c$.

Finally, note that FONC does not guarantee profit maximization. For this, we need, SONC/SOSC:

SONC: $\pi''(y^*) = \frac{\partial MR}{\partial y} - \frac{\partial MC}{\partial y} \leq 0.$

This means that MR must cross MC from above, as the slope of the MR function must be less than the slope of the MC function.

B. Dynamic Choice

B1. Discounting

Suppose you invest \$1 at an annual interest rate $r > 0$. At the end of the year, you will have \$$y = (1+r)$. After two years, you will have $(1+r)^2$. And after t years, you will have $(1+r)^t$. It means that receiving \$1 t years from now is worth at the present time: $\frac{1}{(1+r)^t} = (1+r)^{-t}$. The term $\frac{1}{1+r}$, is called a *discount factor* in *discrete time*. And the *present value* of receiving \$π t years from now is: $\frac{\pi}{(1+r)^t}$.

Now suppose that the bank pays interest at rate r compounded semi-annually. Then every 6 months it pays $\frac{r}{2}$ percent on the amount in the account.

After six months, you receive: $1+\frac{r}{2}$.

After 1 year, you receive: $y = (1+\frac{r}{2})\cdot(1+\frac{r}{2}) = (1+\frac{r}{2})^2$.

Next suppose that the bank pays interest quarterly. Then, after a year, you receive $y = (1+\frac{r}{4})^4$... In general, if the bank pays n times per year, at the end of the year, you receive $y = (1+\frac{r}{n})^n$. As n becomes large, this is called *continuous compounding*. Taking the limit of this expression as $n \to \infty$, we obtain $y = \lim_{n\to\infty}(1+\frac{r}{n})^n = e^r$, where e^r is the *exponential function* of r.

It follows that, under *continuous compounding*, \$1 invested now at an annual interest rate $r > 0$ is worth \$$e^r$ after a year, and \$$e^{rt}$ after t years. Alternatively, \$1 received t years from now is worth at the present time: $\frac{1}{e^{rt}} = e^{-rt}$. Thus, e^{-r} is the *discount factor* in *continuous time*. And the *present value* of receiving \$π t years from now is: $\pi \cdot e^{-rt}$.

Some important properties of the exponential function e^x are:

- if $f(x) = e^x$, then $f'(x) = e^x$.
- if $f(x) = e^{kx}$, then $f'(x) = k \cdot e^{kx}$.

Finally, given $|a| < 1$, note that the infinite sum $S = 1 + a + a^2 + a^3 + \ldots = \frac{1}{1-a}$.

C2. Optimal timing

Let $t = 0$ denote current time. Suppose that an asset grows in value over time, its *market value* at time t being $v(t)$. Suppose that $v'(t) > 0$ and $v''(t) < 0$, so that the asset value grows at a decreasing rate. *Assume that the only value of the asset is its market value $v(t)$.* What is the optimal time to sell the asset to maximize its present value?

Let r be the interest rate. The *present value* of selling the asset at time t and receiving income $v(t)$ at time t is $P(t) = v(t) \cdot e^{-rt}$, where $r > 0$ is the interest rate.

We want to know the optimum selling time t^* that maximizes the present value $P(t)$. Assuming an interior solution, find the value $t^* > 0$ satisfying the FONC:

$$P'(t) = v'(t) \cdot e^{-rt} - r \cdot v(t) \cdot e^{-rt} = 0,$$

or, since $e^{-rt} > 0$,

$$v'(t) = r \cdot v(t),$$

or

$$r = \tfrac{v'(t)}{v(t)}.$$

This implies that, at the optimum selling time $t^* > 0$, the interest rate r must be equal to the growth rate of return $\tfrac{v'(t)}{v(t)}$. Here the interest rate r measures the opportunity cost of money. Then FONC states that, at the margin, *the growth rate of return must be equal to the interest rate*. Note that this applies only to assets that are *not replaced*.

C3. Optimal replacement

Now consider the case where an asset is purchased at time $t = 0$ at a cost $C \geq 0$. Again, the asset grows in value over time, its *market value* at time t being $v(t)$, with $v'(t) > 0$ and $v''(t) < 0$. *Assume that the only value of the asset is its market value* $v(t)$. Then, the *present value of the current asset* sold at time t is

$$\pi(t) = v(t)\, e^{-rt} - C,$$

where $r > 0$ is the interest rate. Also, assume that a new asset is purchased (at a cost C) right after the previous asset is sold at time t for a value $v(t)$. In this context, t denotes the replacement time of the asset.

Example: The case of *forest management* where trees are the asset and their only value is their timber value. The trees are planted (or replanted) on a fixed amount of land every t periods; C is the planting cost; and $v(t)$ is the market value of the trees when they are harvested at time t. In this context, t is the harvest time for trees.

The first asset yields a present value at time $t = 0$ of $\pi(t)$. The second asset (replacing the first one) has a present value $\pi(t) \cdot e^{-rt}$. The third asset (replacing the second one) gives a present value $\pi(t) \cdot e^{-2rt}$, etc…. Assuming an *infinite planning horizon*, it follows that the present value of an infinite succession of assets is

$$P(t) = \pi(t) + \pi(t) \cdot e^{-rt} + \pi(t) \cdot e^{-2rt} + ... = \pi(t)(1 + e^{-rt} + e^{-2rt} + ...) = \tfrac{\pi(t)}{1-e^{-rt}},$$

given that $|e^{-rt}| < 1$, as derived above.

We would like to know what is the optimal replacement time t^* that maximizes the present value $P(t)$. Assuming an interior solution $t^* > 0$, the FONC associated with the maximization of $P(t)$ is

$$P'(t) = 0,$$

or

$$\tfrac{\pi'(t)}{1-e^{-rt}} - \tfrac{r \cdot e^{-rt} \cdot \pi(t)}{(1-e^{-rt})^2} = 0,$$

or

$$\pi'(t) \cdot (1 - e^{-rt}) = r \cdot e^{-rt} \cdot \pi(t),$$

or, given $\pi(t) = v(t) \cdot e^{-rt} - C$,

$$(v'(t) \cdot e^{-rt} - r \cdot e^{-rt} \cdot v(t)) \cdot (1 - e^{-rt}) = r \cdot e^{-rt} \cdot (v(t) \cdot e^{-rt} - C),$$

or, after dividing by e^{-rt} and simplifying,

$$v'(t) \cdot (1 - e^{-rt}) = r \cdot (v(t) - C),$$

or, assuming $v(t) - C > 0$,

$$\frac{v'(t)}{v(t)-C} = \frac{r}{1-e^{-rt}}.$$

The FONC states that, at the optimal replacement time $t^* > 0$, the rate of change in asset value, $\frac{v'(t)}{v(t)-C}$, must equal the opportunity cost of replacement, $\frac{r}{1-e^{-rt}}$. This is called the *Faustmann criterion*. Note that $\frac{r}{1-e^{-rt}} > r$, reflecting that the opportunity cost is higher than the interest rate (because it also includes the opportunity cost of not replanting).