

4.1 PROFIT MAXIMIZATION, N-INPUT, SINGLE OUTPUT

4.1.1 Some basic results

Consider the case of a firm producing one output $y \in \mathbb{R}$ using n inputs $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$. Let $\mathbf{F} \subset \mathbb{R}^{n+1}$ be the feasible set representing the firm technology, where $(y, \mathbf{x}) \in \mathbf{F}$ means that the production decision (y, \mathbf{x}) is technologically feasible. As before, it will be useful to put some structure on the representation of the production technology. Define the production function as

$$g(\mathbf{x}) = \underset{y}{\text{Max}}\{y : (y, \mathbf{x}) \in \mathbf{F}\}.$$

Then, express technological feasibility as

$$\mathbf{F} = \{(y, \mathbf{x}) : y \leq g(\mathbf{x}), \mathbf{x} \geq 0, y \geq 0\}.$$

For a competitive firm in a market economy, let $p \in \mathbb{R}$ be the output price, and $\mathbf{w} = (w_1, w_2, \dots, w_n) \in \mathbb{R}^n$ denote the n input prices for \mathbf{x} . Then, the firm profit is $\pi = p \cdot y - \mathbf{w} \cdot \mathbf{x}$, where $p \cdot y$ is firm revenue, and $\mathbf{w} \cdot \mathbf{x} = \sum_i w_i \cdot x_i$ is firm cost of production. Then, profit maximization is denoted by

$$\underset{y, \mathbf{x}}{\text{Max}}\{p \cdot y - \mathbf{w} \cdot \mathbf{x} : (y, \mathbf{x}) \in \mathbf{F}\}$$

or

$$\underset{y, \mathbf{x}}{\text{Max}}\{p \cdot y - \mathbf{w} \cdot \mathbf{x} : y \leq g(\mathbf{x}), \mathbf{x} \geq 0, y \geq 0\}.$$

Define *technical efficiency* as a production situation (y, \mathbf{x}) where $y = g(\mathbf{x})$. Alternatively, *technical inefficiency* is defined as any situation where $y < g(\mathbf{x})$.

It follows that, given $p > 0$, *profit maximization implies technical efficiency*, where y is chosen to be on the production function $y = g(\mathbf{x})$. Alternatively, any production situation that is technically inefficient (i.e., where $y < g(\mathbf{x})$) must be inconsistent with profit maximization (since it implies the feasibility of increasing revenue without increasing cost).

Note that checking whether a firm is technically efficient or not does not require information about prices (p, \mathbf{w}) . And any observation indicating that a firm is technically inefficient (i.e. that it makes production decisions (y, \mathbf{x}) such that $y < g(\mathbf{x})$) is sufficient to conclude that this firm is not maximizing profit. Two factors can contribute to this situation: 1/ the firm does not have access to the best available technology (suggesting a need to speed up technological adoption for the firm); 2/ the firm management is poor and makes poor use of available resource in the production process (suggesting a need either to replace the management or to improve managerial abilities).

Given $p > 0$ and $y = g(\mathbf{x})$ under technical efficiency, the profit maximization problem can be alternatively written as

$$\underset{\mathbf{x}}{\text{Max}}\{p \cdot g(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x} : \mathbf{x} \geq 0\}.$$

Denote the solution to this optimization problem by the decision rule $\mathbf{x}^*(p, \mathbf{w}) = (x_1^*(p, \mathbf{w}), \dots, x_n^*(p, \mathbf{w}))^T$, the *profit maximizing input demand functions*. Also, given $y = g(\mathbf{x})$, the *profit maximizing output supply function* is $y^*(p, \mathbf{w}) = g(\mathbf{x}^*(p, \mathbf{w}))$. In this section, we have two objectives:

1/ to identify profit maximizing behavior $\mathbf{x}^*(p, \mathbf{w})$ and $y^*(p, \mathbf{w})$;

2/ to investigate the properties of the decision rules $\mathbf{x}^*(p, \mathbf{w})$ and $y^*(p, \mathbf{w})$, providing information on how an economically rational firm would behave under changing markets conditions (as represented by prices p and \mathbf{w}).

4.1.2 PRODUCTION BEHAVIOR

The profit maximization can be written as

$$\text{Max}_{\mathbf{x}} \{ \pi(\mathbf{x}) = R(\mathbf{x}) - C(\mathbf{x}) = p \cdot g(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x} : \mathbf{x} \geq 0 \}.$$

Assume that the function $g(\mathbf{x})$ is twice continuous differentiable. Then, profit maximization is a standard unconstrained optimization problem.

Assume that $\mathbf{x}^*(p, \mathbf{w}) = (x_1^*(p, \mathbf{w}), \dots, x_n^*(p, \mathbf{w}))$ is an *interior solution* to the profit maximization problem. This means that $x_i^*(p, \mathbf{w}) > 0$ for all inputs, $i = 1, \dots, n$.

Given an interior solution, profit maximizing behavior \mathbf{x}^* implies that the first order necessary condition (FONC)

$$\pi'(\mathbf{x}^*) = \left[\frac{\partial \pi(\mathbf{x}^*)}{\partial x_1}, \dots, \frac{\partial \pi(\mathbf{x}^*)}{\partial x_n} \right] = 0,$$

or

$$p \cdot g'(\mathbf{x}^*) - \mathbf{w} = 0,$$

where $g'(\mathbf{x}^*) = \left[\frac{\partial g(\mathbf{x}^*)}{\partial x_1}, \dots, \frac{\partial g(\mathbf{x}^*)}{\partial x_n} \right]$ is a $(1 \times n)$ vector of marginal products, and $\mathbf{w} = (w_1, \dots, w_n)$ is a $(1 \times n)$ vector of input prices. This can be equivalently written as

$$p \cdot g'(\mathbf{x}^*) = \mathbf{w}$$

or

$$p \cdot \frac{\partial g(\mathbf{x}^*)}{\partial x_i} = w_i, i = 1, \dots, n.$$

This is a system of n equations in n unknown, $\mathbf{x}^* = (x_1^*, \dots, x_n^*)^T$. The left-hand side of the equation is marginal revenue $\frac{\partial R}{\partial x_i} = p \cdot \frac{\partial g}{\partial x_i}$. The right-hand side is marginal cost $\frac{\partial C}{\partial x_i} = w_i$. Then, for an interior solution, profit-maximizing behavior means that *marginal revenue must be equal to marginal cost* for each input x_i , $i = 1, \dots, n$.

This has one important implication: when prices are positive ($p > 0$, $w_i > 0$ for all i), the FONC implies that $\frac{\partial g(\mathbf{x}^*)}{\partial x_i} > 0$. This means that *profit-maximizing behavior can only take place at points where the marginal products $\frac{\partial g(\mathbf{x}^*)}{\partial x_i}$ are positive*. Then, profit maximization rules out a priori decision rules where any increase in x_i would contribute to decreasing output. In other words, observing a firm making production decisions (y, \mathbf{x}) in the region where $g(\mathbf{x})$ is declining is sufficient to conclude that this firm does not maximize profit. Again, note that such observation does not require information about prices (p, \mathbf{w}) .

Now, consider the case where resources are scarce. This is expressed by situations where the production technology exhibits *diminishing marginal productivity*. Intuitively, this means that, when marginal product is positive, it becomes increasingly difficult to increase output.

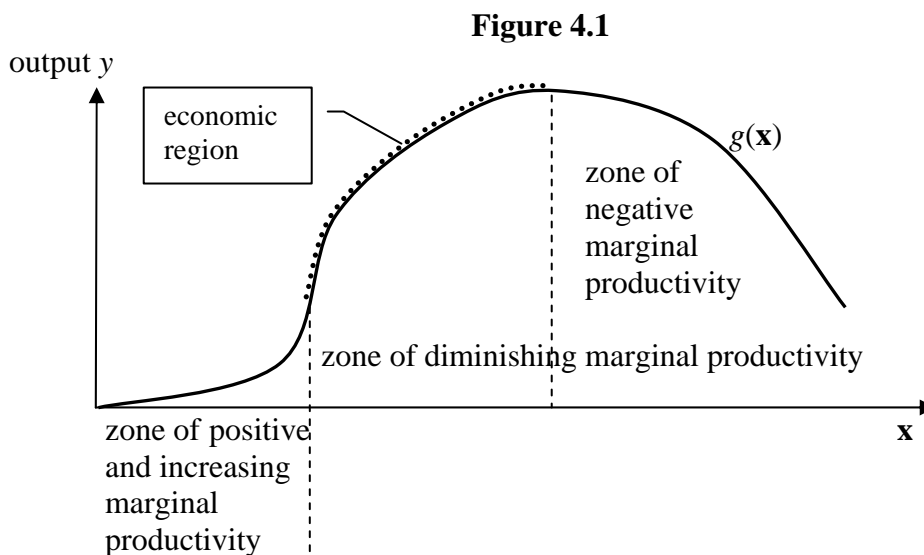
Diminishing marginal productivity is equivalent to the production function $g(\mathbf{x})$ being *concave*, which is also equivalent to $g''(\mathbf{x})$ being a $(n \times n)$ *negative semi-definite* matrix for all \mathbf{x} .

Note that, given $p > 0$, $g''(\mathbf{x}) = \text{negative semi-definite}$ if and only if $\pi''(\mathbf{x}) = p \cdot g''(\mathbf{x}) = \text{negative semi-definite}$ for all \mathbf{x} . It follows that diminishing marginal productivity is equivalent to $\pi''(\mathbf{x})$ being *negative semi-definite* for all \mathbf{x} , which is equivalent to $\pi(\mathbf{x})$ being a *concave* function. And when $\pi(\mathbf{x})$ is a concave function, we know that FONC ($\pi'(\mathbf{x}^*) = 0$) is a necessary and sufficient condition for \mathbf{x}^* to be an interior (global) solution to the profit maximization problem. This gives:

Result 1: Under diminishing marginal productivity, FONC ($\pi'(\mathbf{x}^*) = 0$) is a *necessary and sufficient* condition for \mathbf{x}^* to be a (global) interior solution to the profit maximization problem.

Result 2: Under diminishing marginal productivity, the second order necessary condition SONC ($\pi''(\mathbf{x}) = \text{negative semi-definite}$ at $\mathbf{x} = \mathbf{x}^*$) is always satisfied.

Result 3: At interior points where SONC does *not* hold (meaning where $\pi''(\mathbf{x})$ is *not* negative semi-definite), production decisions *cannot be consistent with profit maximization*. More generally, *profit maximization rules out any interior feasible production region where local concavity (or locally decreasing marginal productivity) does not hold*. Again, such observation does not require having information about prices.



What about the second order sufficient condition (SOSC: $\pi''(\mathbf{x}^*) = \text{negative definite}$)? We have just seen that diminishing marginal productivity implies the negative semi-definiteness of $\pi''(\mathbf{x}^*)$. But $\pi''(\mathbf{x}^*)$ being negative semi-definite does *not* imply that $\pi''(\mathbf{x}^*)$ being negative-definite. It follows that diminishing marginal productivity does *not* imply the SOSC.

Given result 1, it may seem that we may not need the SOSC. This is true if we are just interested in identifying the profit maximizing *level* $\mathbf{x}^*(p, \mathbf{w})$. However, if we also want to investigate the *properties* of $\mathbf{x}^*(p, \mathbf{w})$, then strengthening SONC into SOSC can prove useful.

4.1.3 Properties of the input demand functions $\mathbf{x}^*(p, \mathbf{w})$

We have just shown that diminishing marginal productivity is equivalent to the concavity of the profit function $\pi(\mathbf{x})$. Then, FONC is necessary and sufficient to identify a global interior solution $\mathbf{x}^*(p, \mathbf{w})$ to the profit maximization problem. Yet, this solution may not be unique. And even if it is unique, $\mathbf{x}^*(p, \mathbf{w})$ may not be differentiable functions. In this section, we will consider the convenient case where the decision rules $\mathbf{x}^*(p, \mathbf{w})$ are *differentiable*. This will allow us to make use of calculus tools and to discuss behavioral properties in terms of the derivatives $\frac{\partial \mathbf{x}^*}{\partial p}$ and $\frac{\partial \mathbf{x}^*}{\partial \mathbf{w}}$ measuring the marginal impacts of changes of (p, \mathbf{w}) on profit maximizing behavior \mathbf{x}^* .

As seen above, $\mathbf{x}^*(p, \mathbf{w}) = (x_1^*(p, \mathbf{w}), \dots, x_n^*(p, \mathbf{w}))^\top$ can be interpreted as the solution of the system of equations given by FONC. Denote the profit function by $\pi(\mathbf{x}, p, \mathbf{w}) = p \cdot g(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x}$. Then, FONC is: $\frac{\partial \pi(\mathbf{x}^*, p, \mathbf{w})}{\partial \mathbf{x}} = \pi_{\mathbf{x}}(\mathbf{x}^*, p, \mathbf{w}) = 0$. Apply the *implicit function theorem* to the FONC: Under twice continuous differentiability of $\pi(\mathbf{x}, p, \mathbf{w})$, the solution $\mathbf{x}^*(p, \mathbf{w})$ of the system of equations $\pi_{\mathbf{x}}(\mathbf{x}^*, p, \mathbf{w}) = 0$ is unique and a differentiable function if the $(n \times n)$ matrix $\frac{\partial^2 \pi(\mathbf{x}^*, p, \mathbf{w})}{\partial \mathbf{x}^2} \equiv \pi_{\mathbf{xx}}(\mathbf{x}^*, p, \mathbf{w})$ is invertible, i.e. if the determinant of $\pi_{\mathbf{xx}}(\mathbf{x}^*, p, \mathbf{w})$ is nonzero: $\det(\pi_{\mathbf{xx}}(\mathbf{x}^*, p, \mathbf{w})) \neq 0$.

The implicit function theorem states that $\mathbf{x}^*(p, \mathbf{w})$, the solution to FONC, is a differentiable

function of (p, \mathbf{w}) if $\det(\pi_{\mathbf{xx}}(\mathbf{x}^*, p, \mathbf{w})) \neq 0$, where $\pi_{\mathbf{xx}} \equiv \frac{\partial^2 \pi}{\partial \mathbf{x}^2} \equiv \begin{bmatrix} \frac{\partial^2 \pi}{\partial x_1^2} & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 \pi}{\partial x_1 \partial x_n} \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi}{\partial x_2^2} & \dots & \frac{\partial^2 \pi}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \pi}{\partial x_n \partial x_1} & \frac{\partial^2 \pi}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 \pi}{\partial x_n^2} \end{bmatrix}$ is the

$(n \times n)$ matrix of second derivatives of π with respect to \mathbf{x} . (Note that, under twice continuous differentiability of $\pi(\mathbf{x}, p, \mathbf{w})$, the $(n \times n)$ matrix $\pi_{\mathbf{xx}}$ is *symmetric* from Young theorem). It follows that $\det[\pi_{\mathbf{xx}}(\mathbf{x}^*, p, \mathbf{w})] \neq 0$ is a sufficient condition for $\mathbf{x}^*(p, \mathbf{w})$ to be differentiable functions. We will exploit this condition below.

We start with some insights on the role of the SONC, $\pi_{\mathbf{xx}}(\mathbf{x}^*, p, \mathbf{w}) =$ negative semi-definite, and the SOSC, $\pi_{\mathbf{xx}}(\mathbf{x}^*, p, \mathbf{w}) =$ negative definite. We have the following result:

If $\pi_{\mathbf{xx}} =$ negative semi-definite, then $\det(\pi_{\mathbf{xx}}) \neq 0$ is a *necessary and sufficient* condition to have $\pi_{\mathbf{xx}} =$ negative definite.

(This result is obtained from linear algebra: 1) $\pi_{\mathbf{xx}}$ is a negative (semi) definite matrix if and only if all its eigenvalues are negative (non-positive); 2) the determinant of a matrix is the product of its eigenvalues, implying that a non-zero determinant rules out zero eigenvalues; 3) combining 1) and 2) gives the desired result).

It follows that, if we want to rely on the *implicit function theorem* (with $\det[\pi_{\mathbf{xx}}(\mathbf{x}^*, p, \mathbf{w})] \neq 0$), then the SONC, $\pi_{\mathbf{xx}}(\mathbf{x}^*, p, \mathbf{w}) =$ negative semi-definite, must be strengthened to the SOSC:

$\pi_{\mathbf{xx}}(\mathbf{x}^*, p, \mathbf{w}) =$ negative definite. In this case, strengthening SONC into SOSC will guarantee that $\mathbf{x}^*(p, \mathbf{w})$ are *differentiable* function.

In addition, we have the result:

{SONC together with $\det[\pi_{\mathbf{xx}}(\mathbf{x}^*, p, \mathbf{w})] \neq 0$ } and {SOSC} are *equivalent* statements.

Note: Having $\pi_{\mathbf{xx}}(\mathbf{x}^*, \cdot) =$ negative definite implies that $\pi(\mathbf{x}^*, \cdot)$ is *locally strictly concave* in \mathbf{x} in the close neighborhood of \mathbf{x}^* . However, there exist *strictly concave* functions $\pi(\mathbf{x}, \cdot)$ such that, even though $\pi_{\mathbf{xx}}$ must necessarily be negative semi-definite everywhere, $\pi_{\mathbf{xx}}$ is *not* negative definite at some point (e.g., $\pi(x) = -x^4$ at $x = 0$). This means that SOSC, $\pi_{\mathbf{xx}}(\mathbf{x}^*, p, \mathbf{w}) =$ negative definite, implies that $\pi(\mathbf{x}, p, \mathbf{w})$ must be *locally strictly concave* in the neighborhood of \mathbf{x}^* , although there are strictly concave functions $\pi(\mathbf{x}, p, \mathbf{w})$ that do *not* satisfy SOSC. In that sense, SOSC implies something stronger than local strict concavity...

To proceed with the analysis of profit maximizing behavior, substituting $\mathbf{x}^*(p, \mathbf{w})$ into FONC gives

$$\pi_{\mathbf{x}}(\mathbf{x}^*(p, \mathbf{w}), p, \mathbf{w}) \equiv \frac{\partial \pi(\mathbf{x}^*(p, \mathbf{w}), p, \mathbf{w})}{\partial \mathbf{x}} = 0. \quad (1)$$

4.1.4 The effects of input prices \mathbf{w}

Differentiating FONC in (1) with respect to \mathbf{w} gives (using the chain rule)

$$\frac{\partial^2 \pi}{\partial \mathbf{x}^2} \cdot \frac{\partial \mathbf{x}^*}{\partial \mathbf{w}} + \frac{\partial^2 \pi}{\partial \mathbf{x} \partial \mathbf{w}} = 0, \quad (2a)$$

where $\frac{\partial^2 \pi}{\partial \mathbf{x}^2} = \pi_{\mathbf{xx}} = \begin{bmatrix} \frac{\partial^2 \pi}{\partial x_1^2} & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 \pi}{\partial x_1 \partial x_n} \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi}{\partial x_2^2} & \dots & \frac{\partial^2 \pi}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \pi}{\partial x_n \partial x_1} & \frac{\partial^2 \pi}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 \pi}{\partial x_n^2} \end{bmatrix} = (n \times n) \text{ matrix, } \frac{\partial \mathbf{x}^*}{\partial \mathbf{w}} = \mathbf{X}_{\mathbf{w}}^* = \begin{bmatrix} \frac{\partial x_1^*}{\partial w_1} & \frac{\partial x_1^*}{\partial w_2} & \dots & \frac{\partial x_1^*}{\partial w_n} \\ \frac{\partial x_2^*}{\partial w_1} & \frac{\partial x_2^*}{\partial w_2} & \dots & \frac{\partial x_2^*}{\partial w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n^*}{\partial w_1} & \frac{\partial x_n^*}{\partial w_2} & \dots & \frac{\partial x_n^*}{\partial w_n} \end{bmatrix} =$

$(n \times n) \text{ matrix, and } \frac{\partial^2 \pi}{\partial \mathbf{x} \partial \mathbf{w}} = \pi_{\mathbf{xw}} = \begin{bmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial w_1} & \frac{\partial^2 \pi}{\partial x_1 \partial w_2} & \dots & \frac{\partial^2 \pi}{\partial x_1 \partial w_n} \\ \frac{\partial^2 \pi}{\partial x_2 \partial w_1} & \frac{\partial^2 \pi}{\partial x_2 \partial w_2} & \dots & \frac{\partial^2 \pi}{\partial x_2 \partial w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \pi}{\partial x_n \partial w_1} & \frac{\partial^2 \pi}{\partial x_n \partial w_2} & \dots & \frac{\partial^2 \pi}{\partial x_n \partial w_n} \end{bmatrix} = (n \times n) \text{ matrix, all evaluated at } \mathbf{x}^*(p, \mathbf{w}).$

Under SOSC, we have just seen that $\det[\pi_{\mathbf{xx}}(\mathbf{x}^*, p, \mathbf{w})] \neq 0$ and thus that the $(n \times n)$ matrix $\pi_{\mathbf{xx}}$ is invertible. It follows that (2a) can be written as

$$\frac{\partial \mathbf{x}^*}{\partial \mathbf{w}} \equiv \mathbf{X}_{\mathbf{w}}^* = -[\pi_{\mathbf{xx}}]^{-1} \pi_{\mathbf{xw}}. \quad (2b)$$

This is the *classical comparative static result* showing how profit maximizing input demand would react to changes in input price \mathbf{w} . Note that this result can be obtained simply by applying the implicit function theorem to FONC in (1). Given $\pi(\mathbf{x}, p, \mathbf{w}) = p \cdot g(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x}$, note that $\pi_{\mathbf{xw}} = -\mathbf{I}_n$ is the negative of an identity matrix. Then, equation (2b) becomes particularly informative: it can be written as

$$\frac{\partial \mathbf{x}^*}{\partial \mathbf{w}} \equiv \mathbf{X}_{\mathbf{w}}^* = [\pi_{\mathbf{xx}}]^{-1}. \quad (2c)$$

With $\pi_{\mathbf{x}\mathbf{x}}$ being a symmetric, negative definite matrix (from SOSC), it follows (from linear algebra) that $[\pi_{\mathbf{x}\mathbf{x}}]^{-1}$ is also symmetric, negative definite. This yields the following *important* result:

Given an interior solution and SOSC, profit maximization implies that

$$\frac{\partial \mathbf{x}^*}{\partial \mathbf{w}} \equiv \mathbf{X}_{\mathbf{w}}^* \equiv \begin{bmatrix} \frac{\partial x_1^*}{\partial w_1} & \frac{\partial x_1^*}{\partial w_2} & \dots & \frac{\partial x_1^*}{\partial w_n} \\ \frac{\partial x_2^*}{\partial w_1} & \frac{\partial x_2^*}{\partial w_2} & \dots & \frac{\partial x_2^*}{\partial w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n^*}{\partial w_1} & \frac{\partial x_n^*}{\partial w_2} & \dots & \frac{\partial x_n^*}{\partial w_n} \end{bmatrix} \text{ is a symmetric, negative definite matrix.}$$

This is important for three reasons:

First, this result is obtained under an *arbitrary technology*. In other words, it is a fundamental insight about the general implications of profit maximization behavior.

Second, the diagonal elements of a negative definite matrix are always negative, implying that $\frac{\partial x_i^*}{\partial w_i} < 0$ for all i . This states that increasing an input price always generates a decrease in the quantity demanded for that input. This is a fundamental property of profit maximizing behavior: *profit-maximizing input demand functions are downward sloping*. This is also a very intuitive result: *increasing resource scarcity (as represented by a higher input price) means that profit-maximizing firms have an incentive to reduce the use of the scarce input*.

Finally, the symmetry of the $\mathbf{x}_{\mathbf{w}}^*$ matrix implies that $\frac{\partial x_i^*}{\partial w_j} = \frac{\partial x_j^*}{\partial w_i}$ for all $i \neq j$. These are called *symmetry restrictions*. With n inputs, there are $(n^2-n)/2$ symmetry restrictions. They state that the marginal effect of a change in the i -th input price w_i on the j -th input demand x_j^* must be equal to the marginal effect of a change in the j -th input price w_j on the i -th input demand x_i^* , for $i \neq j$. While they may not be very intuitive, the symmetry restrictions are in fact implied by profit maximizing behavior. This provides useful information on the properties of the decision rules $\mathbf{x}^*(p, \mathbf{w})$, information which can be used in the empirical investigation of production behavior. Also, it means that if the symmetry restrictions were observed *not* to be satisfied, this would be sufficient to conclude that the *observed behavior is inconsistent with profit maximization*. This can provide a basis for the empirical testing of whether actual decision rules are consistent with profit maximization as a representation of economic rationality for production decisions.

Note: In the n -input case, the symmetry, negative semi-definiteness of the matrix $\frac{\partial \mathbf{x}^*}{\partial \mathbf{w}} \equiv \mathbf{X}_{\mathbf{w}}^*$ allows for cross-price effects $\frac{\partial x_i^*}{\partial w_j}$ to be either positive or negative. This will depend on the nature of the underlying technology. More specific results can be obtained in the 2-input case (see below). This issue will be revisited later in the context of cost minimization.

4.1.5 The effects of output price p

Similarly, differentiate FONC in (1) with respect to p to obtain (using the chain rule)

$$\frac{\partial^2 \pi}{\partial \mathbf{x}^2} \cdot \frac{\partial \mathbf{x}^*}{\partial p} + \frac{\partial^2 \pi}{\partial \mathbf{x} \partial p} = 0,$$

or

$$\pi_{\mathbf{xx}} \mathbf{x}_p^* + \pi_{\mathbf{x}p} = 0, \quad (3a)$$

where $\frac{\partial^2 \pi}{\partial \mathbf{x}^2} = \pi_{\mathbf{xx}} = \begin{bmatrix} \frac{\partial^2 \pi}{\partial x_1^2} & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 \pi}{\partial x_1 \partial x_n} \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi}{\partial x_2^2} & \dots & \frac{\partial^2 \pi}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \pi}{\partial x_n \partial x_1} & \frac{\partial^2 \pi}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 \pi}{\partial x_n^2} \end{bmatrix} = (n \times n) \text{ matrix, } \frac{\partial \mathbf{x}^*}{\partial p} = \mathbf{x}_p^* = \begin{bmatrix} \frac{\partial x_1^*}{\partial p} \\ \vdots \\ \frac{\partial x_n^*}{\partial p} \end{bmatrix} = (n \times 1) \text{ vector, and}$

$$\frac{\partial^2 \pi}{\partial \mathbf{x} \partial p} = \pi_{\mathbf{x}p} = \begin{bmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial p} \\ \vdots \\ \frac{\partial^2 \pi}{\partial x_n \partial p} \end{bmatrix} = (n \times 1) \text{ vector, all evaluated at } \mathbf{x} = \mathbf{x}^*(p, \mathbf{w}). \text{ Under SOSOC, we have seen that}$$

$\det[\pi_{\mathbf{xx}}(\mathbf{x}^*, p, \mathbf{w})] \neq 0$ and thus that the $(n \times n)$ matrix $\pi_{\mathbf{xx}}$ is invertible. It follows from (3a) that

$$\frac{\partial \mathbf{x}^*}{\partial p} \equiv \mathbf{x}_p^* = - [\pi_{\mathbf{xx}}]^{-1} \pi_{\mathbf{x}p}. \quad (3b)$$

Note that this result can also be obtained applying the implicit function theorem to FONC. Given

$$\pi(\mathbf{x}, p, \mathbf{w}) = p g(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x}, \text{ we have } \pi_{\mathbf{x}p} = \begin{bmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial p} \\ \vdots \\ \frac{\partial^2 \pi}{\partial x_n \partial p} \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial x_1} \\ \vdots \\ \frac{\partial g}{\partial x_n} \end{bmatrix} = \left(\frac{\partial g}{\partial \mathbf{x}} \right)^T. \text{ Thus, (3b) can be written as}$$

$$\frac{\partial \mathbf{x}^*}{\partial p} \equiv \mathbf{x}_p^* = - [\pi_{\mathbf{xx}}]^{-1} \left(\frac{\partial g}{\partial \mathbf{x}} \right)^T. \quad (3c)$$

In addition, from (3c),

$$\frac{\partial g}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}^*}{\partial p} = - \left(\frac{\partial g}{\partial \mathbf{x}} \right) \cdot [\pi_{\mathbf{xx}}]^{-1} \cdot \left(\frac{\partial g}{\partial \mathbf{x}} \right)^T \geq 0, \quad (3d)$$

since the matrices $\pi_{\mathbf{xx}}$ and $[\pi_{\mathbf{xx}}]^{-1}$ are each negative-definite (from SOSOC),

This has two implications:

- In general, equation (3c) allows for $\frac{\partial x_i^*}{\partial p}$ to be either positive or negative. Thus, the exact effects of output price p on input demand depend on the underlying technology (see Figure 4.1). We will revisit this issue in the context of cost minimization behavior.
- Expression (3d) implies that $\frac{\partial g}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}^*}{\partial p} \geq 0$, i.e. that a weighted average (with $\frac{\partial g}{\partial x_i} > 0$ as weights) of the price slopes $\frac{\partial x_i^*}{\partial p}$ is non-negative.

4.1.6 Properties of the supply function $y^*(p, \mathbf{w})$

We have seen that, given $p > 0$, profit maximizing behavior implies that $y = g(\mathbf{x})$. Thus, the profit maximizing output supply function is

$$y^*(p, \mathbf{w}) = g(\mathbf{x}^*(p, \mathbf{w})) \quad (4)$$

Differentiating equation (4) with respect to p gives (using the chain rule)

$$\begin{aligned}\frac{\partial y^*}{\partial p} &= \frac{\partial g}{\partial x} \cdot \frac{\partial x^*}{\partial p}, \text{ or using (3b),} \\ &= - \frac{\partial g}{\partial x} \cdot [\pi_{xx}]^{-1} \cdot \left(\frac{\partial g}{\partial x}\right)^T \geq 0,\end{aligned}\tag{5}$$

since $[\pi_{xx}]^{-1}$ is a negative-definite matrix from SOSC. This is an important result. It shows that $\frac{\partial y^*}{\partial p} \geq 0$, i.e., that the *profit-maximizing output function tends to be upward sloping*. This is an intuitive finding: *increasing the price of a commodity tends to stimulate its production by a profit-maximizing firm*.

Next, differentiating (4) with respect to w gives

$$\begin{aligned}\frac{\partial y^*}{\partial w} &= \frac{\partial g}{\partial x} \cdot \frac{\partial x^*}{\partial w}, \text{ or using (2c),} \\ &= \frac{\partial g}{\partial x} \cdot [\pi_{xx}]^{-1},\end{aligned}\tag{6a}$$

where $\frac{\partial y^*}{\partial w} = \left(\frac{\partial y^*}{\partial w_1}, \dots, \frac{\partial y^*}{\partial w_n}\right)$ is a $(1 \times n)$ vector.

Comparing (3c) and (6a), it follows that

$$\frac{\partial y^*}{\partial w} = - \left(\frac{\partial x^*}{\partial p}\right)^T\tag{6b}$$

Equation (6b) establishes *n symmetry restrictions* between the slopes of the supply function $\frac{\partial y^*}{\partial w}$ and the negative of the slopes of the input demand functions $-\left(\frac{\partial x^*}{\partial p}\right)^T$: $\frac{\partial y^*}{\partial w_i} = -\frac{\partial x_i^*}{\partial p}$, $i = 1, \dots, n$.

These symmetry restrictions are implied by profit maximization. They can be used in the empirical investigation of production decisions. Alternatively, they can be used to test for profit maximizing behavior in the sense that any empirical evidence against them can be interpreted as evidence that actual decision rules are not consistent with profit maximization.

Finally, from (6b), our earlier discussion related to the properties of $\frac{\partial x^*}{\partial p}$ can be adapted to $\frac{\partial y^*}{\partial w}$ to give the following results:

- In general, $\frac{\partial y^*}{\partial w_i}$ can be either positive or negative. Thus, the exact effects of input price w_i on output supply depend on the underlying technology.
- Using (6a), we have

$$\frac{\partial y^*}{\partial w} \cdot \left(\frac{\partial g}{\partial x}\right)^T = \frac{\partial g}{\partial x} \cdot [\pi_{xx}]^{-1} \cdot \left(\frac{\partial g}{\partial x}\right)^T \leq 0,\tag{6c}$$

implying that a weighted average (with $\frac{\partial g}{\partial x} > 0$ as weights) of the price slopes $\frac{\partial y^*}{\partial w_i}$ is non-positive.

4.1.7 The homogeneity property

Recall our profit maximization problem for a competitive firm

$$\text{Max}_{y, \mathbf{x}} \{ p \cdot y - \mathbf{w} \cdot \mathbf{x} : (y, \mathbf{x}) \in \mathbf{F} \},$$

with $\mathbf{x}^*(p, \mathbf{w})$ as solution. Now, consider the modified problem where all prices are rescaled by a constant $k > 0$,

$$\text{Max}_{y, \mathbf{x}} \{ k \cdot p \cdot y - k \cdot \mathbf{w} \cdot \mathbf{x} : (y, \mathbf{x}) \in \mathbf{F} \},$$

which has for solution $\mathbf{x}^*(kp, k\mathbf{w})$. Note this modified problem can be alternatively written as

$$k \cdot [\text{Max}_{y, \mathbf{x}} \{ p \cdot y - \mathbf{w} \cdot \mathbf{x} : (y, \mathbf{x}) \in \mathbf{F} \}],$$

which has for solution $\mathbf{x}^*(p, \mathbf{w})$. It follows that $\mathbf{x}^*(kp, k\mathbf{w}) = \mathbf{x}^*(p, \mathbf{w})$ for any $k > 0$.

This shows that profit maximizing decision rules $\mathbf{x}^*(p, \mathbf{w})$ are *homogeneous of degree zero in prices* (p, \mathbf{w}). It means that *profit-maximizing behavior is invariant to a proportional change in all prices* (e.g., it is invariant to changing the unit of measurement of money).

This homogeneity property imposes restrictions on behavior. To see that, under the continuous differentiability of $\mathbf{x}^*(p, \mathbf{w})$, apply *Euler theorem* to the functions $\mathbf{x}^*(p, \mathbf{w})$ that are homogeneous of degree zero. This gives:

$$\frac{\partial \mathbf{x}^*}{\partial p} \cdot p + \frac{\partial \mathbf{x}^*}{\partial \mathbf{w}} \cdot \mathbf{w} = 0, \quad (7a)$$

where \mathbf{w} is a $(n \times 1)$ vector, or

$$\frac{\partial x_i^*}{\partial p} \cdot p + \sum_j \frac{\partial x_i^*}{\partial w_j} \cdot w_j = 0, \quad i = 1, \dots, n,$$

or, for $x_i^* > 0$,

$$\frac{\partial x_i^*}{\partial p} \cdot \frac{p}{x_i^*} + \sum_j \frac{\partial x_i^*}{\partial w_j} \cdot \frac{w_j}{x_i^*} = 0, \quad i = 1, \dots, n,$$

or

$$\frac{\partial \ln x_i^*}{\partial \ln p} + \sum_j \frac{\partial \ln x_i^*}{\partial \ln w_j} = 0, \quad i = 1, \dots, n,$$

where $\frac{\partial \ln x_i^*}{\partial \ln p} = \frac{\partial x_i^*}{\partial p} \cdot \frac{p}{x_i^*}$ is the elasticity of x_i^* with respect to output price p , and $\frac{\partial \ln x_i^*}{\partial \ln w_j} = \frac{\partial x_i^*}{\partial w_j} \cdot \frac{w_j}{x_i^*}$ is the elasticity of x_i^* with respect to the j -th input price w_j . (Note: A price elasticity can be conveniently interpreted as measuring the percentage change in a quantity due to a one percent change in price).

It follows that, *for each input demand, the sum of price elasticities across all prices (p, \mathbf{w}) must be zero*. This generates n *homogeneity restrictions* (one for each input demand), as implied by profit maximization.

Finally, with $y^*(p, \mathbf{w}) = g(\mathbf{x}^*(\mathbf{w}, p))$, similar results apply to the output supply function $y^*(p, \mathbf{w})$. The profit maximizing output supply function $y^*(p, \mathbf{w})$ is *homogeneous of degree zero in prices* (p, \mathbf{w}), implying the *additional homogeneity restriction*

$$\frac{\partial y^*}{\partial p} \cdot p + \frac{\partial y^*}{\partial \mathbf{w}} \cdot \mathbf{w} = 0, \quad (7b)$$

where \mathbf{w} is a $(n \times 1)$ vector, or, for $y^* > 0$,

$$\frac{\partial \ln y^*}{\partial \ln p} + \sum_j \frac{\partial \ln y^*}{\partial \ln w_j} = 0,$$

where $\frac{\partial \ln y^*}{\partial \ln p} = \frac{\partial y^*}{\partial p} \cdot \frac{p}{y^*}$ is the elasticity of y^* with respect to output price p , and $\frac{\partial \ln y^*}{\partial \ln w_j} = \frac{\partial y^*}{\partial w_j} \cdot \frac{w_j}{y^*}$ is the elasticity of y^* with respect to the j -th input price w_j .

It follows that *the sum of price elasticities of supply $y^*(p, \mathbf{w})$ across all prices (p, \mathbf{w}) must be 0*.

Note: Given $p > 0$, a simple way to impose the homogeneity restrictions is to write the behavioral rules as follows: $\mathbf{x}^*(p, \mathbf{w}) = \mathbf{x}^*(\frac{\mathbf{w}}{p})$, and $y^*(p, \mathbf{w}) = y^*(\frac{\mathbf{w}}{p})$. Indeed, *using price ratios \mathbf{w}/p is a*

convenient way to impose the restriction that a proportional change in all prices would have no effect on economic behavior.

4.2 THE TWO INPUT CASE (N = 2)

Here, we consider the special case of a two-input firm, where $n = 2$. Of course, all the general results obtained above apply: upward sloping supply function, downward sloping input demand, symmetry restrictions, homogeneity restrictions, etc. However, the two input case will give us some additional results.

When $n = 2$, $\pi'' = \pi_{\mathbf{xx}} = p \cdot g_{\mathbf{xx}}$, where $g_{\mathbf{xx}} = \frac{\partial^2 g}{\partial \mathbf{x}^2} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$, where $g_{ij} \equiv \frac{\partial^2 g}{\partial x_i \partial x_j}$. Under twice

continuous differentiability of $g(\mathbf{x})$, $g_{12} = g_{21}$ (from Young theorem). The SOSC give: $g_{11} < 0$, $g_{22} < 0$, $g_{11} \cdot g_{22} - g_{12}^2 > 0$. And, from linear algebra,

$$\begin{aligned} [\pi_{\mathbf{xx}}]^{-1} &= \frac{1}{p} \cdot [g_{\mathbf{xx}}]^{-1} \\ &= \frac{1}{p \cdot D} \cdot \begin{bmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{bmatrix}, \end{aligned} \quad (8)$$

where $D = g_{11} \cdot g_{22} - g_{12} \cdot g_{12} > 0$ from SOSC. (Check that $\pi_{\mathbf{xx}} [\pi_{\mathbf{xx}}]^{-1} = \text{an identity matrix}$).

Substituting equation (8) into (2c) gives

$$\frac{\partial \mathbf{x}^*}{\partial \mathbf{w}} \equiv \mathbf{x}_{\mathbf{w}}^* = \frac{1}{p \cdot D} \cdot \begin{bmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{bmatrix}, \quad (2c')$$

which implies, given $p > 0$ and $D > 0$ (from SOSC),

$$\frac{\partial x_i^*}{\partial w_j} = \frac{g_{ij}}{p \cdot D} < 0, \quad i \neq j,$$

and

$$\frac{\partial x_i^*}{\partial w_j} = \frac{\partial x_j^*}{\partial w_i} = -\frac{g_{12}}{p \cdot D} = \text{sign}\{-g_{12}\}, \quad i \neq j.$$

This has *two interesting implications*.

- First, interpreting $D = g_{11} \cdot g_{22} - g_{12} \cdot g_{12}$ as measuring the "degree of curvature" of $g(\mathbf{x})$, this suggests that price response $|\frac{\partial x_i^*}{\partial w_j}|$ is larger (smaller) when the production function is "flatter" ("more curved").
- Second, it shows that the cross price effect $\frac{\partial x_i^*}{\partial w_j} = \frac{\partial x_j^*}{\partial w_i}$ is positive (negative) when $g_{12} = \frac{\partial^2 g}{\partial x_1 \partial x_2} < 0$ (> 0), i.e. when x_2 has a negative (positive) impact on the marginal product of x_1 , $\frac{\partial g}{\partial x_1}$.
- This indicates how the nature of the underlying technology affects price response.

4.3 THE GENERAL CASE

So far, we have limited our analysis to the case of a firm producing a single output. We now consider the general case where a multi-output firm produces m outputs using n inputs. For that purpose, it will be convenient to change the notation. Let the decisions made by the firm be represented by the vector of *netputs*, where *outputs* are defined to be *positive* while *inputs* are defined to be *negative*. Let $\mathbf{y} = (y_1, \dots, y_m, y_{m+1}, \dots, y_{m+n}) \in \mathbf{R}^{m+n}$, where (y_1, \dots, y_m) represents m outputs and the n inputs are $(y_{m+1}, \dots, y_{m+n}) (= -(x_1, \dots, x_n)$ in our earlier notation). Let the

underlying technology be represented by the feasible set \mathbf{F} , where $\mathbf{y} \in \mathbf{F}$ means that $\mathbf{y} = (y_1, \dots, y_m, y_{m+1}, \dots, y_{m+n})$ is technologically feasible and satisfies $y_i \geq 0$ for $i = 1, \dots, m$ (stating that the first m elements are *non-negative outputs*) and $y_i \leq 0$ for $i = m+1, \dots, m+n$ (stating that the last n elements are *non-positive inputs*).

In a market economy, let $\mathbf{p} = (p_1, \dots, p_m, p_{m+1}, \dots, p_{m+n}) \in \mathbf{R}^{m+n}$ denote the market prices for the netputs \mathbf{y} . Again, we will assume that prices are positive for all netputs: $p_i > 0$ for all i . It follows that firm revenue is $R = \sum_{i \leq m} p_i y_i$, and that the firm cost is $C = \sum_{i \geq m+1} p_i (-y_i)$, where the negative sign reflects the fact that inputs are defined to be negative. Then, the firm profit is $\pi(\mathbf{y}, \mathbf{p}) = R - C = \mathbf{p} \cdot \mathbf{y} = \sum_i p_i y_i$. Thus a competitive firm maximizing profit would behave so as to solve the optimization problem

$$\underset{\mathbf{y}}{\text{Max}} \{ \mathbf{p} \cdot \mathbf{y} : \mathbf{y} \in \mathbf{F} \}.$$

Let the solution of this maximization problem be $\mathbf{y}^*(\mathbf{p})$. By definition, $\mathbf{y}^*(\mathbf{p})$ must satisfy

$$\mathbf{p} \cdot \mathbf{y}^*(\mathbf{p}) \geq \mathbf{p} \cdot \mathbf{y}, \text{ for all } \mathbf{y} \in \mathbf{F}.$$

Note that, at this point, we have not assumed an interior solution; we have not assumed differentiability; and we have not required that $\mathbf{y}^*(\mathbf{p})$ satisfies FONC, SONC, or SOSC (e.g., we allow for corner solutions).

Consider K observations (e.g., from K different firms in an industry), each observation being associated with market prices \mathbf{p}^k and netput vector \mathbf{y}^k . Then, from the above inequality evaluated at price \mathbf{p}^k , profit maximization becomes

$$\mathbf{p}^k \cdot \mathbf{y}^*(\mathbf{p}^k) \geq \mathbf{p}^k \cdot \mathbf{y}, \text{ for all } \mathbf{y} \in \mathbf{F}.$$

Letting $\mathbf{y}^k = \mathbf{y}^*(\mathbf{p}^k)$ under profit maximization, and choosing $\mathbf{y} = \mathbf{y}^{k'}$, this yields for each k

$$\mathbf{p}^k \cdot \mathbf{y}^k \geq \mathbf{p}^k \cdot \mathbf{y}^{k'}, \text{ for all } k' = 1, \dots, K. \quad (\text{WAPM})$$

This is called the *weak axiom of profit maximization*. Clearly, profit maximization implies WAPM. WAPM provides a simple and convenient way to check whether the k -th firm behaves in a way consistent with profit maximization. If some of the inequalities in WAPM are not satisfied for some (k, k') , then this is sufficient to conclude that the k -th firm does not maximize profit.

Note that, under profit maximization (where $\mathbf{y}^k = \mathbf{y}^*(\mathbf{p}^k)$ for all k), WAPM can be written as

$$\mathbf{p}^k \cdot [\mathbf{y}^*(\mathbf{p}^k) - \mathbf{y}^*(\mathbf{p}^{k'})] \geq 0.$$

Switching \mathbf{p}^k and $\mathbf{p}^{k'}$, this gives

$$-\mathbf{p}^{k'} \cdot [\mathbf{y}^*(\mathbf{p}^k) - \mathbf{y}^*(\mathbf{p}^{k'})] \geq 0.$$

Summing these two inequalities, we obtain

$$[\mathbf{p}^k - \mathbf{p}^{k'}] \cdot [\mathbf{y}^*(\mathbf{p}^k) - \mathbf{y}^*(\mathbf{p}^{k'})] \geq 0,$$

or

$$\sum_i [p_i^k - p_i^{k'}] \cdot [y_i^*(\mathbf{p}^k) - y_i^*(\mathbf{p}^{k'})] \geq 0. \quad (9)$$

Equation (9) indicates that, under profit maximization, increases in prices p from $\mathbf{p}^{k'}$ to \mathbf{p}^k tend to increase netput demand $\mathbf{y}^*(\mathbf{p})$, i.e. that *netput functions* $\mathbf{y}^*(\mathbf{p})$ *tend to be upward sloping*. It shows that this intuitive property is *quite general* (e.g., it applies without differentiability and in the presence of corner solutions).

Next, consider the case where \mathbf{p}^k is in a close neighborhood of $\mathbf{p}^{k'}$ and $\mathbf{y}^*(\mathbf{p})$ is continuously differentiable, then a first order Taylor series expansion of $\mathbf{y}^*(\mathbf{p})$ in the neighborhood of $\mathbf{p}^{k'}$ gives

$$\mathbf{y}^*(\mathbf{p}^k) \cong \mathbf{y}^*(\mathbf{p}^{k'}) + \frac{\partial \mathbf{y}^*(\mathbf{p}^{k'})}{\partial \mathbf{p}} \cdot [\mathbf{p}^k - \mathbf{p}^{k'}],$$

where $[\partial \mathbf{y}^*/\partial \mathbf{p}]$ is a $(m+n) \times (m+n)$ matrix evaluated at $\mathbf{p}^{k'}$, and $[\mathbf{p}^k - \mathbf{p}^{k'}]$ is a $(m+n) \times 1$ vector. Substituting this expression into (9) yields

$$[\mathbf{p}^k - \mathbf{p}^{k'}]^T \cdot \frac{\partial \mathbf{y}^*(\mathbf{p}^{k'})}{\partial \mathbf{p}} \cdot [\mathbf{p}^k - \mathbf{p}^{k'}] \geq 0.$$

Since this is true for all \mathbf{p}^k chosen in a close neighborhood of $\mathbf{p}^{k'}$, this implies that $\frac{\partial \mathbf{y}^*}{\partial \mathbf{p}}$ is a $(m+n) \times (m+n)$ positive semi-definite matrix.

Combining this with the symmetry restrictions derived earlier, we obtain the following key result

$$\frac{\partial \mathbf{y}^*}{\partial \mathbf{p}} = \text{symmetric, positive semi-definite matrix.} \quad (10a)$$

This is a multi-output generalization of earlier results about the behavioral implications of profit maximization. It gives the following general results.

- *supply functions tend to be upward-sloping*: $\frac{\partial y_i^*}{\partial p_i} \geq 0, i = 1, \dots, m.$
- *input demand functions tend to be downward-sloping*: $\frac{\partial (-y_i^*)}{\partial p_i} \leq 0, i = m+1, \dots, m+n.$
- *the symmetry restrictions apply*: $\frac{\partial y_i^*}{\partial p_j} = \frac{\partial y_j^*}{\partial p_i},$ for all netputs $i \neq j.$

In addition, we have seen the profit maximization decision rules $\mathbf{y}^*(\mathbf{p})$ are homogeneous of degree zero in prices p . Using *Euler theorem*, this gives the *homogeneity restrictions*

$$\sum_j \frac{\partial y_j^*}{\partial p_j} \cdot p_j = 0, i = 1, \dots, n+m,$$

or, using elasticities,

$$\sum_j \frac{\partial \ln y_j^*}{\partial \ln p_j} = 0, i = 1, \dots, n+m,$$

or, using matrix notation,

$$\frac{\partial \mathbf{y}^*}{\partial \mathbf{p}} \cdot \mathbf{p} = 0, \quad (10b)$$

where $\frac{\partial \mathbf{y}^*}{\partial \mathbf{p}}$ is a $(m+n) \times (m+n)$ matrix, and \mathbf{p} is a $(m+n) \times 1$ vector.

The homogeneity restrictions (10b) imply that the columns of the matrix $\frac{\partial \mathbf{y}^*}{\partial \mathbf{p}}$ are linearly dependent. This linear dependency is inconsistent with $\frac{\partial \mathbf{y}^*}{\partial \mathbf{p}}$ being positive-definite. It follows that the homogeneity restrictions imply that the matrix $\frac{\partial \mathbf{y}^*}{\partial \mathbf{p}}$ is *only positive semi-definite*.

4.4 THE LECHATLIER PRINCIPLE

Consider the case where the netput vector \mathbf{y} is partitioned into two parts: $\mathbf{y} = (\mathbf{y}_a, \mathbf{y}_b).$

Accordingly, partition the corresponding price vector: $\mathbf{p} = (\mathbf{p}_a, \mathbf{p}_b).$ Then, profit maximization gives

$$\text{Max}_{\mathbf{y}_a, \mathbf{y}_b} \{ \mathbf{p}_a \cdot \mathbf{y}_a + \mathbf{p}_b \cdot \mathbf{y}_b : (\mathbf{y}_a, \mathbf{y}_b) \in \mathbf{F} \} \quad (11a)$$

giving the profit maximizing decisions $\mathbf{y}^* = (\mathbf{y}_a, \mathbf{y}_b) = (\mathbf{y}_a^*(\mathbf{p}_a, \mathbf{p}_b), \mathbf{y}_b^*(\mathbf{p}_a, \mathbf{p}_b)).$

Now consider the case where the firm is *constrained to choose only the netputs* \mathbf{y}_a . This could happen under two scenarios:

- Scenario 1: it may be that the firm chooses netput \mathbf{y}_a more frequently than \mathbf{y}_b . In this context, \mathbf{y}_a are characterized as *short term decisions* (e.g., including variable inputs) while \mathbf{y}_b are *long term decisions* (e.g., capital).
- Scenario 2: it may be that the firm faces regulatory constraints that dictate the firm choice for netputs \mathbf{y}_b .

In either scenario, the profit-maximizing firm would choose \mathbf{y}_a as follows

$$\text{Max}_{\mathbf{y}_a} \{ \mathbf{p}_a \cdot \mathbf{y}_a + \mathbf{p}_b \cdot \mathbf{y}_b : (\mathbf{y}_a, \mathbf{y}_b) \in \mathbf{F} \} = \mathbf{p}_b \cdot \mathbf{y}_b + \text{Max}_{\mathbf{y}_a} \{ \mathbf{p}_a \cdot \mathbf{y}_a : (\mathbf{y}_a, \mathbf{y}_b) \in \mathbf{F} \} \quad (11b)$$

giving the choices $\mathbf{y}_a^c(\mathbf{p}_a, \mathbf{y}_b)$. Since these choices take the value of \mathbf{y}_b as given, the decision rules $\mathbf{y}_a^c(\mathbf{p}_a, \mathbf{y}_b)$ in (11b) are *conditional on* \mathbf{y}_b . They contrast with the decisions rules $\mathbf{y}_a^*(\mathbf{p}_a, \mathbf{p}_b)$ in (11a) that are *unconditional*. The question is: what is the relationship between the conditional and unconditional rules $\mathbf{y}_a^c(\mathbf{p}_a, \mathbf{y}_b)$ and $\mathbf{y}_a^*(\mathbf{p}_a, \mathbf{p}_b)$?

To answer that question, consider the simple case where we evaluate \mathbf{y}_b in (11b) at $\mathbf{y}_b = \mathbf{y}_b^*(\mathbf{p}_a, \mathbf{p}_b)$. Then, the profits obtained in (11a) and (11b) are necessarily the same. However, economic behavior (as represented by the two decision rules $\mathbf{y}_a^c(\mathbf{p}_a, \mathbf{y}_b)$ and $\mathbf{y}_a^*(\mathbf{p}_a, \mathbf{p}_b)$) is not the same. To examine how they differ, consider the relationships between $\mathbf{y}_a^*(\mathbf{p}_a, \mathbf{p}_b)$ and $\mathbf{y}_a^c(\mathbf{p}_a, \mathbf{y}_b)$, evaluated at $\mathbf{y}_b = \mathbf{y}_b^*(\mathbf{p}_a, \mathbf{p}_b)$:

$$\mathbf{y}_a^*(\mathbf{p}_a, \mathbf{p}_b) \equiv \mathbf{y}_a^c(\mathbf{p}_a, \mathbf{y}_b^*(\mathbf{p}_a, \mathbf{p}_b)). \quad (12)$$

Differentiating (12) with respect to \mathbf{p}_a and \mathbf{p}_b yields

$$\frac{\partial \mathbf{y}_a^*}{\partial \mathbf{p}_a} = \frac{\partial \mathbf{y}_a^c}{\partial \mathbf{p}_a} + \frac{\partial \mathbf{y}_a^c}{\partial \mathbf{y}_b} \cdot \frac{\partial \mathbf{y}_b^*}{\partial \mathbf{p}_a}, \quad (13a)$$

and

$$\frac{\partial \mathbf{y}_a^*}{\partial \mathbf{p}_b} = \frac{\partial \mathbf{y}_a^c}{\partial \mathbf{y}_b} \cdot \frac{\partial \mathbf{y}_b^*}{\partial \mathbf{p}_b}. \quad (13b)$$

Assuming that the matrix $\frac{\partial \mathbf{y}_b^*}{\partial \mathbf{p}_b}$ is invertible, (13b) implies

$$\frac{\partial \mathbf{y}_a^c}{\partial \mathbf{y}_b} = \frac{\partial \mathbf{y}_a^*}{\partial \mathbf{p}_b} \cdot \left(\frac{\partial \mathbf{y}_b^*}{\partial \mathbf{p}_b} \right)^{-1}.$$

Substituting this expression into (13a) yields

$$\frac{\partial \mathbf{y}_a^*}{\partial \mathbf{p}_a} = \frac{\partial \mathbf{y}_a^c}{\partial \mathbf{p}_a} + \frac{\partial \mathbf{y}_a^*}{\partial \mathbf{p}_b} \cdot \left(\frac{\partial \mathbf{y}_b^*}{\partial \mathbf{p}_b} \right)^{-1} \cdot \frac{\partial \mathbf{y}_b^*}{\partial \mathbf{p}_a},$$

or, using the symmetry restriction from (10a), $\frac{\partial \mathbf{y}_b^*}{\partial \mathbf{p}_a} = \left(\frac{\partial \mathbf{y}_a^*}{\partial \mathbf{p}_b} \right)^T$,

$$\frac{\partial \mathbf{y}_a^*}{\partial \mathbf{p}_a} - \frac{\partial \mathbf{y}_a^c}{\partial \mathbf{p}_a} = \frac{\partial \mathbf{y}_a^*}{\partial \mathbf{p}_b} \cdot \left(\frac{\partial \mathbf{y}_b^*}{\partial \mathbf{p}_b} \right)^{-1} \cdot \left(\frac{\partial \mathbf{y}_a^*}{\partial \mathbf{p}_b} \right)^T. \quad (14)$$

The LeChatelier principle:

$$\frac{\partial \mathbf{y}_a^*}{\partial \mathbf{p}_a} - \frac{\partial \mathbf{y}_a^c}{\partial \mathbf{p}_a} = \text{symmetric, positive semi-definite matrix,}$$

showing that the unconditional price effects $\frac{\partial \mathbf{y}_a^*}{\partial \mathbf{p}_a}$ exceed the conditional price effects $\frac{\partial \mathbf{y}_a^c}{\partial \mathbf{p}_a}$ by a positive semi-definite matrix.

Proof: From (10a), $\frac{\partial y_b^*}{\partial p_b}$ is a symmetric, positive semi-definite matrix. Being invertible (by assumption), it must be symmetric positive definite, and its inverse $(\frac{\partial y_b^*}{\partial p_b})^{-1}$ is also symmetric positive definite. It follows that the right-hand side of (14) is a symmetric, positive semi-definite matrix. Thus, the left-hand side must also be a symmetric, positive semi-definite matrix.

The LeChatelier principle implies that

$$\frac{\partial y_{ai}^*}{\partial p_{ai}} - \frac{\partial y_{ai}^c}{\partial p_{ai}} \geq 0,$$

or

$$\frac{\partial y_{ai}^*}{\partial p_{ai}} \geq \frac{\partial y_{ai}^c}{\partial p_{ai}} \geq 0, \text{ or to avoid ambiguity } \left| \frac{\partial y_{ai}^*}{\partial p_{ai}} \right| \geq \left| \frac{\partial y_{ai}^c}{\partial p_{ai}} \right| \geq 0.$$

This states that unconditional price effects are larger than conditional price effects. For profit maximizing firms, this can be interpreted in two ways:

- In scenario 1, it means that long-term decisions are more price responsive than short-term decisions. Alternatively, it suggests that *economic adjustments to changing market prices can be expected to be smaller in the short term compared to the longer term.*
- In scenario 2, it means that *regulatory constraints reduce the ability of firms to respond to changing market prices.*