# AAE 635 Lecture 2

## 2.1 REVIEW

[Profit-maximizing firm's problem, and FONC, SOSCmax…]

## 2.2 THE ONE-INPUT MAXIMIZATION PROBLEM, CONTINUED.

Recall that this problem involves a revenue function *R(x)*, and input cost *wx*. The objective is to maximize profits. We developed a <u>numerical</u> example in which the input demand function is indeed downward sloping; note that this input demand function was extracted from the FONC:

$$Max_x\ [200\,x - 4\,x^2 - w \cdot x] \tag{2.1}$$

$$200 - 8x\ - w = 0$$

$$8\,x = 200 - w, \text{ i.e. } \quad x^* = 25 - w/8.$$

We seek a way generalize this result. What must be true about the firm's problem to assure that demand is downward sloping? Answer: satisfaction of the necessary and sufficient conditions.

### 2.2.1 ANALYTICAL SOLUTION

Expand the revenue function to differentiate between the production function *g* and the output price *p*:

$$\underset{x}{Max}\ \ \pi(x), \text{ or } \underset{x}{Max}\ \ \{p \cdot g(x) - w \cdot x\} \tag{2.2}$$

Then the FONC is …

$$p \cdot g'(x) - w = 0 \tag{2.3}$$

or $p \cdot g'(x) = w$.

(Economic intuition: value of the marginal product, VMP = marginal input cost)

Note that, when both prices are positive ($w > 0$ and $p > 0$), this implies that $g'(x^*) > 0$. This means that profit maximizing inputs must correspond to points of *positive marginal productivity*, where additional units of *x* increase output *y*.

And our SONC is…

$$p \cdot g''(x) \le 0, \text{ or } g''(x) \le 0 \text{ (if } p > 0). \tag{2.4}$$

(Intuition: additional units of *x* increase output, but at a non-increasing rate)

Finally, our SOSC is…

$$g''(x) < 0. \tag{2.5}$$

This means *diminishing marginal productivity*: additional units of *x* must increase output at a decreasing rate.

Now suppose the FONC can be solved for the value of $x^*$ in terms of the parameters of the problem. For instance, if $g(x) = a\,x + b\,x^2$, then we have

$$\underset{x}{Max} \quad \{p \cdot (ax + bx^2 - w \cdot x\} \qquad (2.6)$$

with FONC

$$p \cdot (a + 2bx) - w = 0. \qquad (2.7)$$

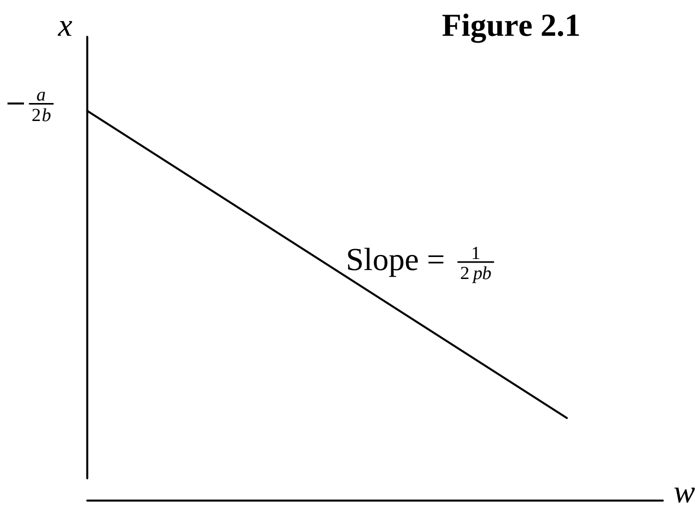
This can be solved for $x^*$ in terms of $p$ and $w$:

$$pa + 2pbx = w \qquad (2.8)$$

$$2pbx = w - pa$$

$$x^* = \frac{w}{2pb} - \frac{a}{2b}. \qquad (2.9)$$

**Figure 2.1**




It follows that $\frac{\partial x^*}{\partial w} = \frac{1}{2pb} < 0$ under the SOSC. Again, this shows that the input demand function is downward sloping.

### 2.2.2 GENERAL CASE

In the general case with an interior solution, consider the FONC (2.3) and SOSC (2.5). Let $x^*(p, w)$ denote the solution to the profit maximization problem (2.2). Substituting $x^*(p, w)$ into the FONC gives the following <u>identity</u>

$$p\,g'(x^*(p, w)) - w \equiv 0. \qquad (2.10)$$

Next, differentiate both sides of this identity with respect to *w* (using the chain rule) to obtain

$$pg''(x^*)\,\frac{\partial x^*}{\partial w} - 1 = 0 .$$

Given *p* > 0 and under the SOSC (*g"(x*)* < 0), this yields

$$\frac{\partial x^*}{\partial w} = \frac{1}{pg''(x^*)} < 0. \qquad (2.11)$$

Again, this implies that a *profit maximizing input demand function is downward sloping* (where input demand decreases with a rise in the input price). This result holds under general technology and general market conditions. This shows that such a property is a generic implication of profit maximizing behavior. It can be the subject of empirical investigation and/or test. This means that one would expect to observe downward sloping demand functions for inputs when profit motives are strong. Alternatively, if a demand function for input were observed *not* to be downward sloping, this would imply that production decisions are not made in a way consistent with profit maximization.

The results are summarized in the next table.

| Function | Numerical | Example | Analytical |
|---|---|---|---|
| Objective function $\pi$ | $200x - 4x^2 - wx$ | $p[ax + bx^2] - wx$ | $p\,g(x) - wx$ |
| Production function | $(200x - 4x^2)/p$ | $ax + bx^2$ | $g(x)$ |
| FONC | $(200 - 8x) - w = 0$ | $p[a + 2bx] - w = 0$ | $p\,g'(x) - w = 0$ |
| SOSC | $-8 < 0$ | $2pb < 0$ | $p\,g''(x) < 0$ |
| Input demand | $x^* = 25 - w/8$ | $x^* = \frac{w}{2pb} - \frac{a}{2b}$ | $x^*(p, w)$ |
| Slope of input demand function | $\frac{\partial x^*}{\partial w} = -1/8$ | $\frac{\partial x^*}{\partial w} = \frac{1}{2pb}$ | $\frac{\partial x^*}{\partial w} < 0$ |

Note that the "numerical" case is obtained from the "example" with *a* = 200 and *b* = - 4.

### 2.2.3 ADDITIONAL ANALYTICAL RESULTS

Given fluctuating market conditions, it is of interest to analyze the effects of changing market prices *p* and *w* on the production decisions made by a competitive firm.

#### A. Effects of output price *p* on input demand

Differentiate both sides of the identity (2.10) with respect to *p* to obtain

$$g'(x^*) + p\,g''(x^*)\,\frac{\partial x^*}{\partial p} = 0.$$

Given positive prices (*p* > 0, *w* > 0) and under the SOSC (*g"(x*)* < 0), this yields

$$\frac{\partial x^*}{\partial p} = -\frac{g'(x^*)}{pg''(x^*)} > 0,$$

since $g'(x^*) > 0$ from FOC. This shows that increasing output price $p$ tends to increase the demand for input $x$ under profit maximization.

**B. Effects of prices on output supply**

Under profit maximization, output is given by the production function $y = g(x^*)$. Then, consider the profit maximizing output supply function

$$y^*(p, w) = g(x^*(p, w)).$$

Differentiating this expression with respect to $p$ and $w$ (using the chain rule) gives

$$\frac{\partial y^*}{\partial p} = g'(x^*)\,\frac{\partial x^*}{\partial p},$$

and

$$\frac{\partial y^*}{\partial w} = g'(x^*)\,\frac{\partial x^*}{\partial w}.$$

But, we have seen that $g'(x^*) > 0$ under positive prices (from FONC), and that $\frac{\partial x^*}{\partial w} < 0$ and $\frac{\partial x^*}{\partial p} > 0$ under profit maximization. It follows that

$\frac{\partial y^*}{\partial p} > 0$ (*upward sloping output supply function*)

and

$\frac{\partial y^*}{\partial w} < 0$ (negative effect of input price on output supply).

Again, these are generic implications of profit maximization for interior input choice: increasing output price tends to stimulate supply; and increasing input price tends to reduce output supply. They show that economic rationality generates specific implications for production behavior. Such implications can be used either to help explain economic behavior or to help identify situations where profit opportunities are not exploited.

But how general are these results? They were obtained in the context of a competitive firm choosing a single input. There is a need to examine whether they would hold under more general conditions.