AAE 635 Lectures 1&2

1.1 What is Microeconomics?

Economics is the study of allocating limited resources to satisfy unlimited wants.
This tension implies tradeoffs among competing goals. The analysis can be carried out at the micro level (at the level of firms making production decisions, or of households making consumption decisions) or at the aggregate level (market, sector or economy).

There are several key questions:

1. Understanding how production and consumption decisions are made. This is done by investigating actual decision rules. Economic decision rules involve the mapping of the economic environment of decision makers into their decisions. They tell us how changes in the economic environment (reflecting technology, resource scarcity, regulations, government policy, etc.) affect production and consumption decisions.

2. Evaluating whether firms or households are making the best possible decisions. Making the best possible decisions at the micro level defines economic rationality. It is often represented as a constrained optimization problem. For firms, this typically involves profit maximization as a characterization of economic rationality for production decisions. This is the essence of production theory. For households, this typically involves utility maximization subject to a budget constraint as a representation of economic rationality for consumption decisions. This is the essence of consumer theory.

3. Evaluating whether production and consumption decisions are the best that can be done at the aggregate level (society or "the economy"). Making the best possible decisions at the aggregate level defines economic efficiency. Again, this can be expressed as a constrained optimization problem. There are scenarios where microeconomic rationality and economic efficiency are consistent with each other. But there are also situations where they are not. Studying such situations give insights on how to improve the efficiency of resource allocation for both private and public decision makers. The social usefulness of economic analysis derives in large part from such efficiency gains (which are seen through productivity gains, improved standard of living and/or economic development).

Our goal is to formalize economic rationality and economic efficiency. This involves developing formal models of production and consumption allocations. Are such formal models really needed? Mathematical rigor helps generate refutable hypotheses about economic behavior in social science. And it helps entangle economic relationships that are often complex. Both give economics a special advantage over less formal modes of understanding human behavior.

But economic models typically involve simplifying assumptions. As such, they cannot give an accurate of all economic behavior. This raises the question: how to interpret possible discrepancies between a model prediction and observable behavior?

- In *positive* economic analysis, the intent is to explain and predict human behavior. Then discrepancies between observations and theoretical predictions can be used to help refine our conceptual models and/or our measurements.

- In *normative* economic analysis, the intent is to help improve economic decisions. This applies to private decisions (e.g., production, consumption, or investment decisions; contract design) as well as public decisions (e.g., government policies). Then discrepancies between current decisions and efficient decisions mean that that there is a value to making economic recommendations to private or public agents.

Developing the skills needed for conducting such analyses requires a platform of theoretical constructs that we will review. That is the objective of AAE 635.

Microeconomics: the analysis of consumer and firm behavior as represented by constrained optimizations problems, along with an interpretation of the welfare consequences of the decisions made. Thus, the syllabus:
1. Optimization as a representation of economic rationality for firms and consumers;
2. Duality (convenient tools for analyzing economic behavior);
3. Welfare (investigating the welfare consequences of economic decisions for normative analysis);
4. General equilibrium analysis: applying microeconomics at the aggregate level to analyze a sector or an economy.

1.2 NEOCLASSICAL THEORY OF THE FIRM

Consider a firm making production decisions under a given technology. For simplicity, we will start with the case where there is a single output y produced using a single input x. Both the quantity of output produced and the quantity of input used are measured using real numbers: $x \in \mathbf{R}$, $y \in \mathbf{R}$, where $\mathbf{R}$ denotes the set of real numbers.

Current technology imposes restrictions on the feasibility of production decisions. The technologically feasible set for (y, x) is denoted by $\mathbf{F}$, a subset of $\mathbf{R}\times\mathbf{R}$, where feasibility is written as $(y, x) \in \mathbf{F}$. It will be convenient to put some structure on what is technically feasible. First, input choice x and output choice y typically cannot be negative: $x \geq 0$ and $y \geq 0$, or $x \in \mathbf{R}_+$ and $y \in \mathbf{R}_+$ where $\mathbf{R}_+ = \{z: z \geq 0, z \in \mathbf{R}\}$ is the set of non-negative real numbers. Second, technological feasibility puts some upper bound on how much output can be produced for any given input x. This can be represented by the production function (also called production frontier):

$$g(x) = max_y \{y: (y, x) \in \mathbf{F}\}.$$

The production function $g(x)$ converts input x into the largest possible output y.

Combining these two arguments, it follows that technological feasibility can be written as

$$\mathbf{F} = \{(y, x): y \leq g(x), (y, x) \in R_+^2\}, \text{ or}$$

$$\mathbf{F} = \{(y, x): y \leq g(x), x \geq 0, y \geq 0\}.$$

Thus, the firm technology can be represented either by the feasible set $\mathbf{F}$, or by the production function $g(x)$ along with non-negativity restrictions for x and y.

Elements of this:

- Technology: represented by the production possibility frontier $y = g(x)$.
- Decisions. The firm makes production decisions by choosing a point among all the feasible points. This choice depends on the objective of the decision maker.

For simplicity, assume the world is

- Deterministic (there is perfect information about prices and technology)
- Static

In a market economy, using input x involves a monetary cost C. And producing output y generates a monetary revenue R. Then, the firm profit is $\pi = R - C$. Profit is the amount of money generated by the firm after the input cost is paid. A firm cannot choose all variables. If competitive, it is a "price taker": inputs prices w and output price p as given.

Profit maximization

Assume that the firm manager can claim this profit (e.g., the case where the firm manager is also the firm owner). Also, assume that he/she has preferences that are "non-satiated in income", meaning that higher income levels make him/her better off. This non-satiation assumption appears intuitive and reasonable. It implies that the firm manager has an incentive to maximize profit. This is the behavioral premise of economic rationality in production theory.

Consider a competitive firm facing the production technology $\mathbf{F} = \{(y, x): y \leq g(x), x \geq 0, y \geq 0\}$. By definition, a firm is competitive if its decisions cannot influence market prices. The firm profit is $\pi = p \cdot y - w \cdot x$, where $p \cdot y$ is revenue, and $w \cdot x$ is production cost. Then, subject to technical feasibility $y \leq g(x)$, a profit maximizing firm would choose x and y as follows:

$$\underset{x,y}{Max} \ \ p \cdot y - w \cdot x, \quad \text{s.t. } y \leq g(x)$$

or, if $p > 0$,

$$\underset{x}{Max} \ \ p \cdot g(x) - w \cdot x.$$

This maximization problem is a mathematical formulation of the behavioral premise of production theory under competition. It has the disadvantage of not being closely linked with more intuitive or more descriptive understanding of production decisions. However, it has several advantages. The mathematical approach confers analytical tractability that can be extended to the many inputs/many outputs case. It makes it easier to check rigor; it is concise; it may generate insights about behavior or efficiency that are beyond the reach of intuitive reasoning; and it is the current standard of advanced economic knowledge.

1.3 MATHEMATICAL ANALYSES OF FIRM DECISION

Consider a firm that uses 1 input, x, to produce an output y, generating revenue

$$R(x) = 200x - 4x^2 \ \ (= p \cdot g(x) \text{ for a competitive firm}).$$

And costs are: $w \cdot x$

Now we ask: how does the firm respond to a change in the input price w?

To answer this question, we must first present the firm's decision. Here we assume that the firm maximizes profits $\pi(x, w) = R(x) - w \cdot x$. Then, the input choice is formally presented as follows:

$$\underset{x}{Max}\ \{\pi(x,w) : x \geq 0\} \qquad \text{“choose } x \geq 0 \text{ so as to maximize } \pi\text{”}$$

$$= \underset{x}{Max}\ \{R(x) - w \cdot x : x \geq 0\}$$

How do we find the optimal quantity of x that maximizes profit (call it x^*)? Clearly, x^* is the solution to the profit maximization problem. Finding this solution is easier if we assume that profit $\pi(x, \cdot)$ (and thus revenue $R(x, \cdot)$) is a differentiable function of x.

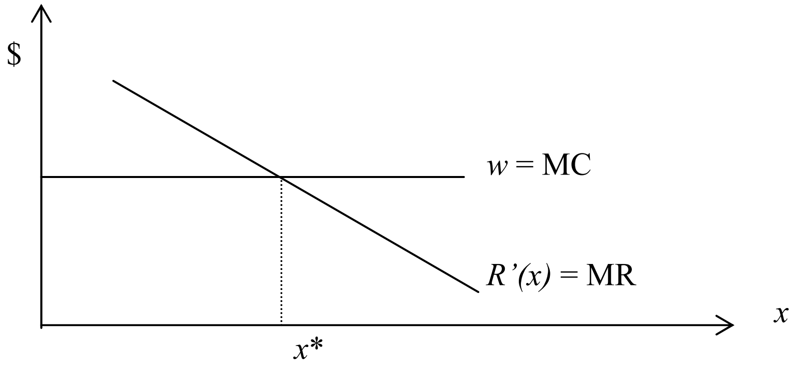
Definition: A function $f(x)$ is *differentiable* at x_0 if the limit $\lim_{x\to x_0} [(f(x)-f(x_0))/(x-x_0): x \neq x_0]$ exists. This limit is the derivative of $f(x)$ at x_0, denoted by $f'(x_0)$ or $f_x(x_0)$, or $\partial f/\partial x(x_0)$. *A function is differentiable if it is differentiable at all possible points* x_0.

Geometrically, the derivative of $f(x)$ at x_0 is the slope of the linear function tangent to $f(x)$ at x_0.

Given that the solution is interior, i.e. $x > 0$, a necessary condition for the solution is that $\partial\pi/\partial x = 0$ (alternatively, $\pi'(x^*) = 0$, or $\pi_x(x^*) = 0$), i.e. that: $R'(x) - w = 0$.

Note that $R'(x)$ is marginal revenue (MR) of input, and w is marginal cost (MC) of input. Then, a necessary condition for the solution is: MR = MC.

Figure 1.1



Now we can solve the firm's problem for the level of x that maximizes profits:

Since $R'(x) = 200 - 8x$, we know that

$$200 - 8x = w,$$

yielding the profit maximizing input demand

$$x^*(w) = 25 - w/8$$

So as long as we know w, we can calculate the profit maximizing value of x. Notice that the demand curve $x^*(w)$ is downward sloping: $\partial x/\partial w = -1/8 < 0$.

Next we want to generalize this result; under what conditions can we be sure that a competitive firm producing its output with a single input will have a downward-sloping input demand curve?

1.4 Digression on Necessary and Sufficient conditions

A. Necessary conditions

1. What is a necessary condition?
2. Why is $\partial\pi/\partial x = 0$ *necessary* at x^*?
3. What's the big deal about an "interior" solution?

1. What is a necessary condition?

A necessary condition is a fact implied by an observation.

Equivalent statements: **A** implies **B**; **A** ⇒ **B**; if **A** then **B**. ***B*** *is necessary for* ***A***.

Example 1: **A**: I live in Madison.
B: I live in Wisconsin. **A ⇒ B.**

Example 2: **A**: x^* solves the revenue maximization problem, $x^*>0$.
B: $\partial\pi/\partial x = 0$. **A ⇒ B.**

Compare sufficient condition: A fact that implies an observation/result.

A *is sufficient for* ***B***: If I live in Madison, then I must live in Wisconsin.

Is **B** sufficient for **A** (does **B** ⇒ **A**)? If I live in Wisconsin, must I live in Madison? No.

Note: "Is **B** sufficient for **A**?" is the same as "is **A** necessary for **B**?"
A ⇒ **B** can be read: **B** is necessary for **A**
A is sufficient for **B**.

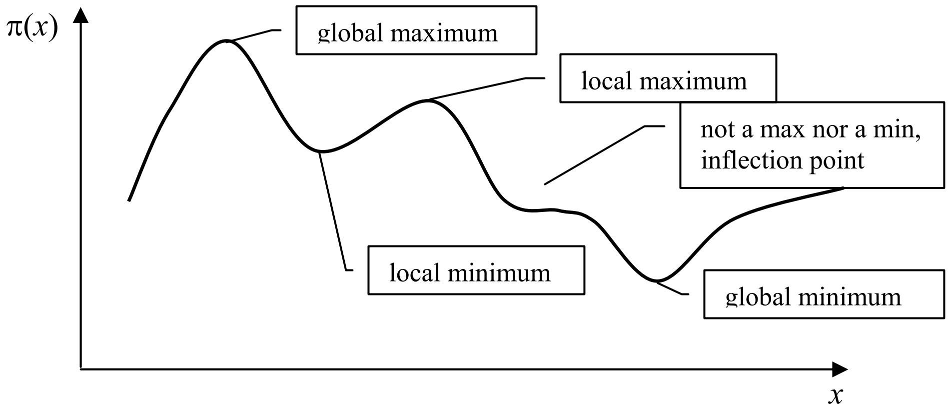
In the Example 2 above, **B** is necessary for **A**; is **B** is sufficient for **A**? That is, if we have some x^* such that $\partial\pi/\partial x = 0$, does x^* solve the problem? We'll answer this shortly.

2. Why is $\partial\pi/\partial x = 0$ *necessary* at x^*?

See Figure 1.2. If the solution is interior ($x^*>0$), then $\partial\pi/\partial x = 0$ must be true. We can now see that *sufficiency* doesn't hold, since a profit function might have several stationary points, some of which could be minima.

Figure 1.2 shows that if π has several extrema, the condition $\pi'(x^*) = 0$ is a *necessary* but not a *sufficient* condition for profit maximization.

Figure 1.2



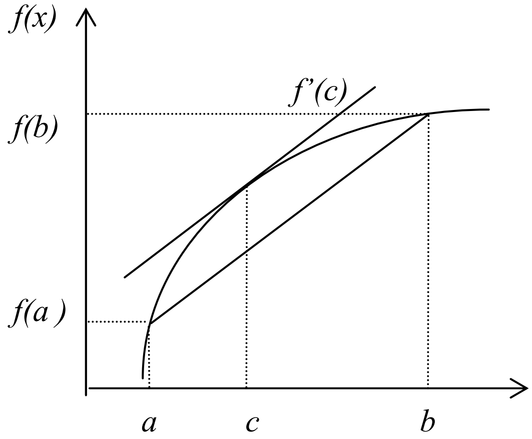
To prove that $\partial\pi/\partial x = 0$ is necessary for a maximum, we need facts (a) and (b), and then can show (c):

(a) If $f(x)$ is everywhere differentiable; and
(b) If x^* is a local maximum, $f(x^*) \geq f(x)$ "in a neighborhood",
(c) Then $f'(x^*) = 0$.

Mean Value Theorem (Silberberg and Suen, p.31):
The slope of a chord joining two points (a, $f(a)$) and (b, $f(b)$) has slope: $[f(b) - f(a)]/(b-a)$.
If f is differentiable, there exists a point c satisfying $a \leq c \leq b$, such that a tangent to the function f at c has the same slope as that of the chord, $f'(c) = [f(b) - f(a)]/(b-a)$.
Thus

$$f(b) = f(a) + (b - a)f'(c) \tag{1.1}$$

Figure 1.3



We use the mean value theorem to prove necessity of $f'(x) = 0$ at a maximum.

<u>Proof</u>: Assume that f is differentiable (thus continuous). Consider two distinct points $x > 0$ and $x_0 > 0$. By the mean value theorem, there exists a x_1 between x_0 and x such that

$$f(x_0) - f(x) = (x_0 - x) \cdot f'(x_1). \tag{1.2}$$

Notice (1.2) is just (1.1) with $b = x_0$, $a = x$, $c = x_1$. This applies for all feasible x and x_0. Assume that $x_0 > 0$ maximizes $f(x)$, implying that $f(x_0) \geq f(x)$ for all $x > 0$. It follows from (1.2) that $(x_0 - x) \cdot f'(x_1) \geq 0$, yielding

$$f'(x_1) \geq 0 \text{ if } x_0 > x,$$
$$f'(x_1) \leq 0 \text{ if } x_0 < x.$$

Choose x in the close neighborhood of x_0. It follows that $f'(x_0) = 0$. In words, the derivative of the function $f(x)$ must be zero at the maximum. The same result would apply to a minimization problem. This proves that $f'(x_0) = 0$ is a necessary condition for x_0 to be local interior maximum to the function $f(x)$.

Finally, since a global maximum is always a local maximum, it follows that:
$f'(x^*) = 0$ is a necessary condition for x^* to be a global interior maximum to $f(x)$.

In general, given an optimization problem for a differentiable function $f(x)$, $f'(x^*) = 0$ is called the *first-order necessary condition* (FONC) for an interior optimum x^*.

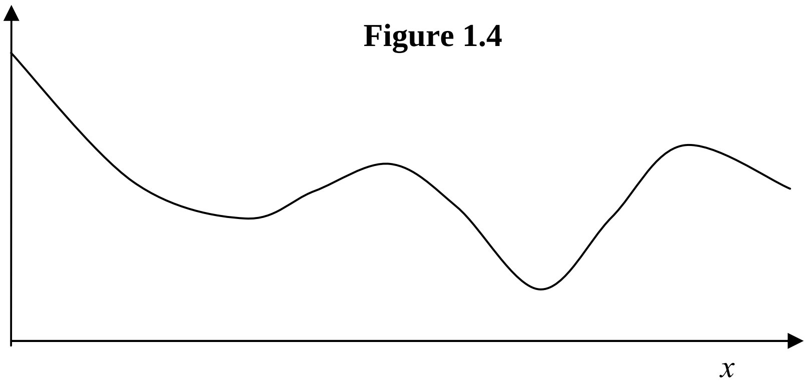
<u>3. What's the big deal about an "interior" solution?</u>

Consider Figure 1.4. The solution is non-interior (i.e. a "corner solution"), so we do not expect $\pi'(x^*) = 0$).

To address such cases, a more general statement of "first order necessary conditions" takes the form of a set of *Kuhn-Tucker* conditions:

(i) $x \geq 0$;
(ii) $f'(x) \leq 0$;
(iii) $x \cdot f'(x) = 0$.

Figure 1.4



Note that, if $x^* > 0$, the above Kuhn-Tucker conditions reduce to the FONC as above.

B. Sufficient conditions

Is the converse true? No, because we have found examples where $\pi'(x) = 0$ when x is not the solution to maximizing $\pi(x)$ (see Figure 1.2). Thus, if our concern is to know whether or not a particular value x^* is truly a maximum, the FONC is not enough. We need something stronger (sufficiency) conditions.

SECOND ORDER NECESSARY CONDITION (SONC)

Consider Figure 1.5. It is clear that both an interior maximum and an interior minimum are characterized by FONC $f'(x^*) = 0$ or $\pi'(x^*) = 0$. It may be apparent that there is a second order necessary condition associated with a maximum.

Figure 1.5

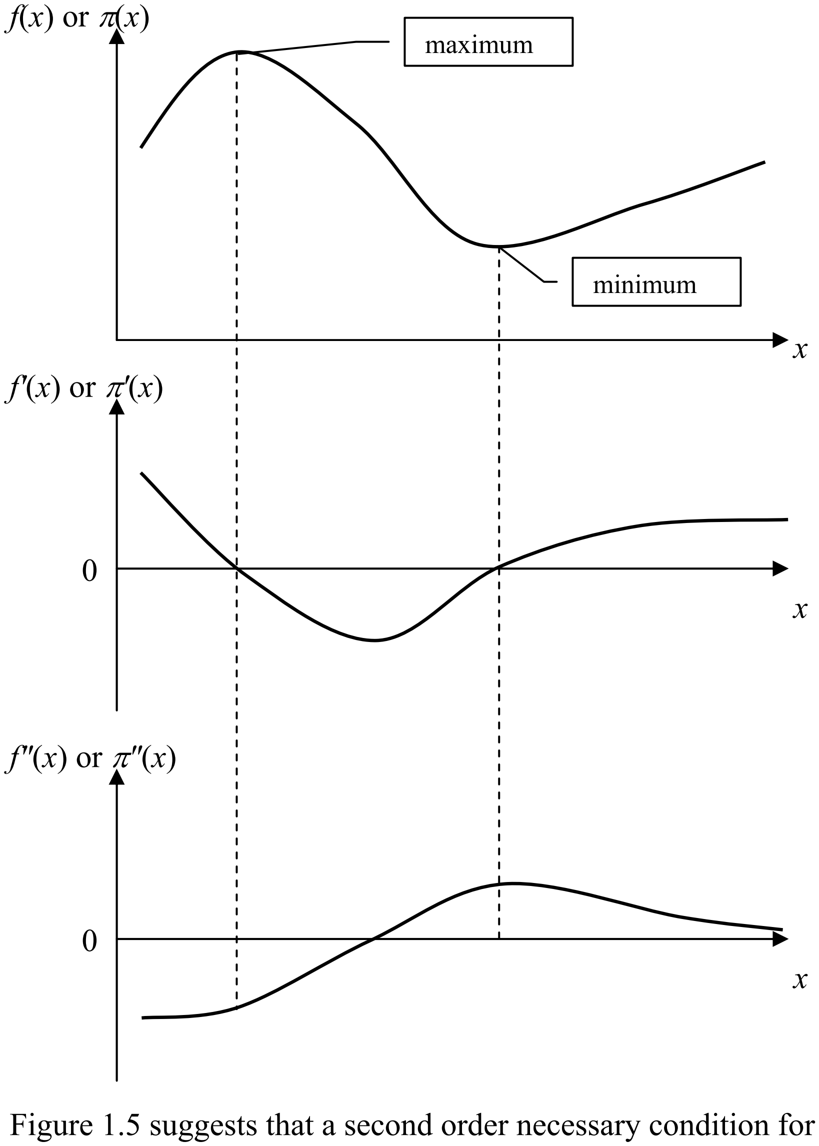


Figure 1.5 suggests that a second order necessary condition for an interior maximum of $f(x)$ is: $f''(x) \leq 0$. Indeed, in general, necessary conditions for an *interior maximum* x^* of $f(x)$ are: $f'(x) = 0$ and $f''(x) \leq 0$ at x^*.

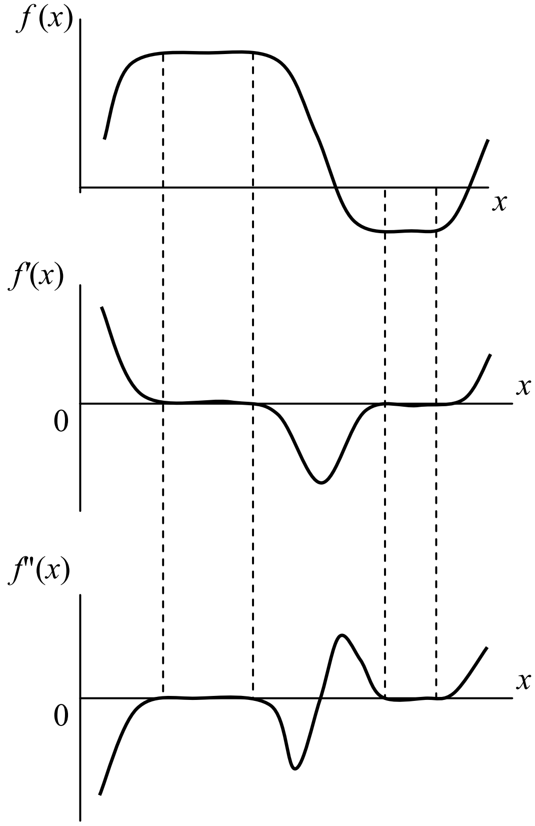
Similarly, Figure 1.5 suggests that a second order necessary condition for an *interior minimum* of $f(x)$ is: $f''(x) \geq 0$. In general, necessary conditions for an *interior minimum* x^* of $f(x)$ are: $f'(x) = 0$ and $f''(x) \geq 0$ at x^*.

Question: Why don't we assert that $f''(x) < 0$ (> 0) is a necessary condition for an interior maximum (minimum) of $f(x)$?

As shown in Figure 1.6, the top (or bottom) of the hill might be flat. Then, there are many maxima (and many minima), with $f'(x) = 0$ and $f''(x) = 0$ for all.

In summary, $f'(x^*) = 0$ and $f''(x^*) \leq 0$ (≥ 0) are necessary conditions for x^* to be an interior solution to the maximization (minimization) of $f(x)$.

Figure 1.6



SECOND ORDER SUFFICIENT CONDITIONS (SOSC)

Is there a set of conditions such that, if we can find an x^* that satisfies them, we know x^* is a maximum of $f(x)$?

Assume that the function $f(x)$ is twice differentiable. Graph the derivative $f'(x)$. From Figure 1.7, the slope of this curve is the second derivative $\partial[\partial f/\partial x]/\partial x = \partial^2 f(x)/\partial x^2 = f_{xx} = f''(x)$. Check the slope of f''. Note that $f''(x^*) < 0$ at the maximum x^* in Figure 1.7.

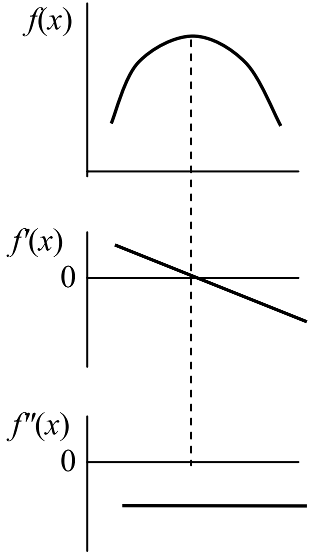
In general, $f'(x^*) = 0$ and $f''(x^*) < 0$ are sufficient conditions for x^* to be a local interior solution to maximizing $f(x)$:

> When $f(x)$ is twice differentiable in the neighborhood of an interior point x^*, if
> $f'(x^*) = 0$ (first-order necessary condition, FONC) , and
> $f''(x^*) < 0$, (second-order sufficient condition, SOSC)
> then x^* is a local interior maximum of $f(x)$ in the neighborhood of x^*.

Similar results apply to a minimization problem. Indeed, $f'(x^*) = 0$ and $f''(x^*) > 0$ are sufficient conditions for x^* to a local interior solution to minimizing $f(x)$:

> When $f(x)$ is twice differentiable in the neighborhood of an interior point x^*, if
> $f'(x^*) = 0$ (first-order necessary condition, FONC), and
> $f''(x^*) > 0$, (second-order sufficient condition, SOSCmin)
> then x^* is a local minimum of $f(x)$ in the neighborhood of x^*.

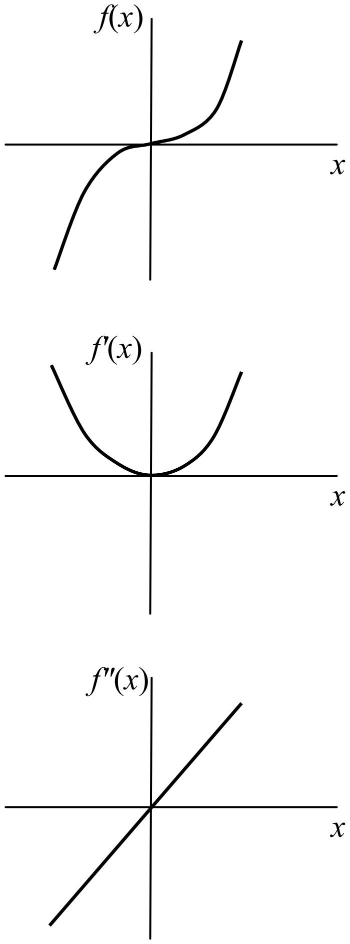
Figure 1.7



<u>Question</u>: Why do we need strict inequality in SOSC (or SOSCmin)?

Consider Figure 1.8, where there is an <u>inflection point</u> satisfying $f'(x^*) = 0$ and $f''(x^*) = 0$. But this inflection point is neither a maximum nor a minimum (either locally or globally). This shows that the necessary conditions FONC-SONC, $f'(x^*) = 0$ and $f''(x^*) \leq 0$ (≥ 0), are not sufficient to identify x^* as a local interior solution to the maximization (minimization) of $f(x)$.

Figure 1.8



Example: Let $\pi(x) = x^3$.

Then, $\pi' = \partial\pi/\partial x = 3x^2 = 0$ at $x = 0$,

And $\pi'' = \partial^2\pi/\partial x^2 = 6x = 0$ at $x = 0$.

Yet, $x = 0$ is not a (local or global) maximum of $\pi(x)$. And it is not a (local or global) minimum of $\pi(x)$.

SUMMARY

In summary, for an interior solution to the maximization of a twice-differentiable function $f(x)$, we have the generally applicable conditions:

FONC	$f'(x^*) = 0$
SONC	$f''(x^*) \leq 0$
SOSC	$f''(x^*) < 0$.

In general, FONC and SONC are necessary (but not sufficient) conditions for x^* to be an interior maximum of $f(x)$. And FONC and SOSC are sufficient to identify x^* as a local interior maximum of $f(x)$. However, without additional information, we cannot be sure that x^* is also a global maximum.

And for an interior solution to the minimization of a twice-differentiable function *f(x)*, we have the generally applicable conditions:

FONC $f'(x^*) = 0$
SONCmin $f''(x^*) \geq 0$
SOSCmin $f''(x^*) > 0$.

Again, FONC and SONCmin are necessary (but not sufficient) conditions for x^* to be an interior minimum of $f(x)$. And FONC and SOSCmin are sufficient to identify x^* as a local interior minimum. However, without additional information, we cannot be sure that x^* is also a global minimum.