

AAE 635

Applied Microeconomic Theory

Fall 2011

10/3/2011

Class 9

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Last class

- Constrained optimization (lecture note 5)
 - Mathematic tools
 - Lagrangean approach
 - FONC & SOC

Today and next class...

- Homework #2 due
- Midterm?
- Constrained optimization (lecture note 5)
 - Optimization under quasi-concavity
- Cost minimization (lecture note 6)

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Lagrangean approach

- General maximization problem with m constraints

$$\underset{\mathbf{x}}{\text{Max}} \{f(\mathbf{x}) : h_1(\mathbf{x}) = 0, \dots, h_m(\mathbf{x}) = 0; \mathbf{x} \geq 0, \mathbf{x} \in \mathbf{R}^n\}$$

- The Lagrangean

$$L(\mathbf{x}, \boldsymbol{\lambda}) \equiv f(\mathbf{x}) + \boldsymbol{\lambda}^T \mathbf{h}(\mathbf{x}) \equiv f(\mathbf{x}) + \sum_j \lambda_j h_j(\mathbf{x})$$

- FONC:

$$\frac{\partial L}{\partial \mathbf{x}} \equiv L_{\mathbf{x}}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \equiv f_{\mathbf{x}}(\mathbf{x}^*) + (\boldsymbol{\lambda}^*)^T \mathbf{h}_{\mathbf{x}}(\mathbf{x}^*) = 0$$

$$\frac{\partial L}{\partial \boldsymbol{\lambda}} \equiv L_{\boldsymbol{\lambda}}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \equiv \mathbf{h}(\mathbf{x}^*) = 0$$

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- SONC:

$$[\mathbf{s}_b(\mathbf{x}_b^*)^T, \mathbf{I}_{n-m}] L_{\mathbf{xx}}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \begin{bmatrix} \mathbf{s}_b(\mathbf{x}_b^*) \\ \mathbf{I}_{n-m} \end{bmatrix}$$

symmetric, negative semi-definite

- SOSC:

$$[\mathbf{s}_b(\mathbf{x}_b^*)^T, \mathbf{I}_{n-m}] L_{\mathbf{xx}}(\mathbf{x}^*, \boldsymbol{\lambda}^*) \begin{bmatrix} \mathbf{s}_b(\mathbf{x}_b^*) \\ \mathbf{I}_{n-m} \end{bmatrix}$$

symmetric, negative definite

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Optimization under quasi-concavity

- If objective function $f(\mathbf{x})$ is quasi-concave, &
 - Constraint functions $\mathbf{h}(\mathbf{x})$ are quasi-concave, &
 - $f_x(\mathbf{x}^*) \neq 0$, &
 - $\lambda^* \geq 0$,
- Lagrangean FONCs are also sufficient for a global interior maximum.

OR

- If objective function $f(\mathbf{x})$ is concave, &
 - Constraint functions $\mathbf{h}(\mathbf{x})$ are quasi-concave, &
 - $\lambda^* \geq 0$,
- Lagrangean FONCs are also sufficient for a global interior maximum.

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In summary: constrained optimization

- Set up the Lagrange equation
- For an interior maximum
 - Is CQ satisfied? (yes, keep going; no, stop)
 - FONC → solve for \mathbf{x}^* (and λ^*)
 - Is SONC (negative semi-definiteness) satisfied? (if no, stop)
 - Check SOSC (negative definiteness)
 - Quasi-concavity and global interior solution
- For an interior minimum
 - Is CQ satisfied? (yes, keep going; if no, stop)
 - FONC → solve for \mathbf{x}^* (and λ^*)
 - Is SONC (positive semi-definiteness) satisfied? (if no, stop)
 - Check SOSC (positive definiteness)
 - Quasi-convexity and global interior solution

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Application: Cost minimization, n -input, 1 output (lecture note 6)

- *Step I:* Set up the problem: $\text{Min}_{\mathbf{x}} \{ \mathbf{w} \cdot \mathbf{x} : y \leq g(\mathbf{x}), \mathbf{x} \geq 0 \}$

- Cost min implies tech. eff. if $\mathbf{w} > 0$: $y = g(\mathbf{x})$
- Firm's problem is rewritten as

$$\text{Min}_{\mathbf{x}} \{ \mathbf{w} \cdot \mathbf{x} : y - g(\mathbf{x}) = 0; \mathbf{x} \geq 0 \}$$

- CQ (with only one constraint): for at least one input i ,

$$\det\left(\frac{\partial \mathbf{h}(\mathbf{x}^c)}{\partial x_i}\right) \equiv \frac{\partial \mathbf{h}(\mathbf{x}^c)}{\partial x_i} \equiv -\frac{\partial g(\mathbf{x}^c)}{\partial x_i} \neq 0$$

- Cost min implies $MP_i > 0$ if $x_i > 0$

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Step II: Find the interior solution: Lagrangean approach

- Write the Lagrangean equation:

$$L(\mathbf{x}, \lambda) \equiv \mathbf{w} \cdot \mathbf{x} + \lambda(y - g(\mathbf{x}))$$

- Given CQ satisfied, solve FONC to obtain $\mathbf{x}^c(y, \mathbf{w})$:

$$L_x(\mathbf{x}^c, \lambda^c) = \mathbf{w}^T - \lambda^c g_x(\mathbf{x}^c) = 0$$

$$L_\lambda(\mathbf{x}^c, \lambda^c) = y - g(\mathbf{x}^c) = 0$$

- Economic interpretation: $MRTS_{ij} = \frac{\frac{\partial g(\mathbf{x}^c)}{\partial x_i}}{\frac{\partial g(\mathbf{x}^c)}{\partial x_j}} = \frac{w_i}{w_j}$
- $\lambda^c > 0$, constraint is binding
- In graph...

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