AAE 635

Applied Microeconomic Theory

Fall 2009

Last class

- Cost minimization problem

Today & Next class (and more)

- Homework 3 due Oct. 20th
- Midterm in class on Oct. 22th
- Cost minimization
  - Two inputs and general case
- Duality (lecture note 8, chapters 3,5&6)
- Cost minimization (continued) (lecture note 9, chapter 3,5&6)

Math Roadmap II

Duality

- Primal approach: FONC's
- Dual approach: via indirect objective function
- The problem: $\underset{\mathbf{x}}{Max}\ \{f(\mathbf{x},\boldsymbol{\alpha}):\mathbf{h}(\mathbf{x},\boldsymbol{\alpha})=0,\mathbf{x}\geq 0,\mathbf{x}\in\mathbf{R}^n\}$
  - Direct objective fn = $f(\mathbf{x},\alpha)$
  - Indirect objective fn= $f(\mathbf{x}^*(\alpha),\alpha) = V(\alpha)$
- Envelop theorem: $\frac{\partial V}{\partial \boldsymbol{\alpha}} \equiv V_{\boldsymbol{\alpha}} = L_{\boldsymbol{\alpha}}(\mathbf{x}^*(\boldsymbol{\alpha}),\boldsymbol{\lambda}^*(\boldsymbol{\alpha}),\boldsymbol{\alpha})$
- Interpreting λ

The primal-dual approach

- Alternative SOSC: the bordered Hessian

$$\mathbf{H} \equiv \frac{\partial^2 L}{\partial(\mathbf{x},\boldsymbol{\lambda})^2} \equiv \begin{bmatrix} L_{\mathbf{xx}} & (\mathbf{h}_{\mathbf{x}})^T \\ \mathbf{h}_{\mathbf{x}} & 0 \end{bmatrix} \text{ being negative definite,}$$

$$\text{s.t. } \mathbf{h}_{\mathbf{x}} \cdot \begin{bmatrix} \mathbf{v}_a \\ \mathbf{v}_b \end{bmatrix} = 0,\ \mathbf{v}_b \neq 0$$

- The primal-dual results (CQ, interior solution):

$$V_{\boldsymbol{\alpha\alpha}} - L_{\boldsymbol{\alpha\alpha}}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\alpha}) = [L_{\boldsymbol{\alpha}\mathbf{x}}(\mathbf{x}^*, \boldsymbol{\lambda}^*), L_{\boldsymbol{\alpha\lambda}}(\mathbf{x}^*, \boldsymbol{\lambda}^*)] \begin{bmatrix} \mathbf{x}^*_{\boldsymbol{\alpha}} \\ \boldsymbol{\lambda}^*_{\boldsymbol{\alpha}} \end{bmatrix},$$

symmetric and positive semi-definite

Special cases

- Linear objective function: $f(\mathbf{x},\alpha) = \alpha^T \cdot \mathbf{x}$
- Constraint functions do not contain α: $\mathbf{h}(\mathbf{x}) = 0$

- The primal-dual results (CQ, interior solution):

$$V_{\boldsymbol{\alpha\alpha}} - L_{\boldsymbol{\alpha\alpha}}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\alpha}) = V_{\boldsymbol{\alpha\alpha}} = [\mathbf{I}_n, 0] \begin{bmatrix} \mathbf{x}^*_{\boldsymbol{\alpha}} \\ \boldsymbol{\lambda}^*_{\boldsymbol{\alpha}} \end{bmatrix} = \mathbf{x}^*_{\boldsymbol{\alpha}}$$

Symmetric, positive semi-definite matrix

Implications

- $V(\alpha)$ is homogeneous of degree one and convex in α (why? because $V_{\alpha\alpha}$ is p.s.d…)
  - Convexity being a general property of $V(\alpha)$ for any linear obj. fn (does not rely on interior solution…)
- **x** is homogeneous of degree zero in α
- $\mathbf{x}_\alpha$ is symmetric and p.s.d (why?)
- What about minimization?

Profit maximization

- Indirect profit function $V(\alpha)$
- Envelop theorem ➔ Hotelling Lemma
  - The profit maximizing netput demand functions are obtained via the derivative of indirect profit function with respect to the corresponding price
- Netput demand functions are homogeneous of degree zero in prices
- $\mathbf{x}_\alpha$ is symmetric and p.s.d ➔ upward sloping output supply and downward sloping input demand, and symmetric restrictions

Cost minimization

- Indirect cost function $V(\alpha)$
- Envelop theorem ➔ Shephard Lemma
  - The cost minimizing input demand functions are obtained via the derivative of indirect cost function with respect to the corresponding price
- Input demand functions are homogeneous of degree zero in prices
- $\mathbf{x}_\alpha^c$ is symmetric and n.s.d ➔ downward sloping input demand and symmetric restrictions