AAE 635

Applied Microeconomic Theory

Fall 2009

Last class

- Constrained optimization
  - Unconstrained approach
  - Lagrange approach

Today & Next class (and more)

- Homework 2 due Oct. 6th
- Homework 3 out Oct. 8th, due 20th
- Midterm (pick up a date!)
- Cost minimization (lecture note 7, ch. 4 &5)
- Duality (lecture note 8, chapter 6)

Review: constrained optimization

- Set up the Langrange equation
- For an interior maximum
  - Is CQ satisfied? (yes, keep going; no, stop)
  - FONC ➔ solve for **x*** (and **λ***)
  - Is SONC (negative semi-definiteness) satisfied? (if no, stop)
  - Check SOSC (negative definiteness)
  - Quasi-concavity and global interior solution

- For an interior minimum
  - Is CQ satisfied? (yes, keep going; if no, stop)
  - FONC ➔ solve for **x*** (and **λ***)
  - Is SONC (positive semi-definiteness) satisfied? (if no, stop)
  - Check SOSC (positive definiteness)
  - Quasi-convexity and global interior solution

Cost minimization, *n*-input, 1 output

- The problem:

$$\underset{\mathbf{x}}{Min}\ \{\mathbf{w}\cdot\mathbf{x} : y \le g(\mathbf{x}), \mathbf{x} \ge 0\}$$

- Cost min implies tech. eff.: $y=g(\mathbf{x})$
- Firm’s problem is rewritten as

$$\underset{\mathbf{x}}{Min}\ \{\mathbf{w}\cdot\mathbf{x} : y - g(\mathbf{x}) = 0; \mathbf{x} \ge 0\}$$

- CQ (with only one constraint): for at least one input *i*,

$$\frac{\partial \mathbf{h}(\mathbf{x}^c)}{\partial x_i} \equiv -\frac{\partial g(\mathbf{x}^c)}{\partial x_i} \ne 0$$

Production Behavior, assuming interior solutions…

$$\underset{\mathbf{x}}{Min}\ \{\mathbf{w}\cdot\mathbf{x} : y - g(\mathbf{x}) = 0; \mathbf{x} \geq 0\}$$

- The Lagrangean: $L(\mathbf{x},\boldsymbol{\lambda}) \equiv \mathbf{w}\cdot\mathbf{x} + \lambda(y - g(\mathbf{x}))$
- Solve FONC to obtain $\mathbf{x}^c(y, \mathbf{w})$:

$$L_x(\mathbf{x}^c, \lambda^c) = \mathbf{w}^T - \lambda^c g_x(\mathbf{x}^c) = 0$$
$$L_\lambda(\mathbf{x}^c, \lambda^c) = y - g(\mathbf{x}^c) = 0$$

- Economic interpretation: $MRTS_{ij} = \dfrac{\frac{\partial g(\mathbf{x}^c)}{\partial x_i}}{\frac{\partial g(\mathbf{x}^c)}{\partial x_j}} = \dfrac{w_i}{w_j}$
- $\lambda^c > 0$, constraint is binding

If $g(\mathbf{x})$ is quasi-concave

- Concave production function (diminishing MP)?

Then,

- FONC is also sufficient
- $\text{SONC}_{\min}$ (positive semi-def.) always satisfied
  - If $\text{SONC}_{\min}$ not satisfied, then production cannot be cost minimizing
- Does not imply $\text{SOSC}_{\min}$ (positive def.)
  - Strengthen to $\text{SOSC}_{\min}$ to do comparative statics

Input demand functions $\mathbf{x}^c(y, \mathbf{w})$, to assume differentiable, we need

- Strengthen SONC to SOSC, ➔ the bordered Hessian $\mathbf{H} \equiv \frac{\partial^2 L}{\partial(\mathbf{x}, \lambda)^2} \equiv \begin{bmatrix} L_{\mathbf{xx}} & L_{\mathbf{x}\lambda} \\ L_{\lambda\mathbf{x}} & L_{\lambda\lambda} \end{bmatrix}$ being non-singular
- SOSC ➔ det[**H**] $\neq 0$, and $g(\mathbf{x})$ locally strictly quasi-concave
- FONC identity $L_x(\mathbf{x}^c(y, \mathbf{w}), \lambda^c(y, \mathbf{w}), \mathbf{w}, y) \equiv 0$

  $L_\lambda(\mathbf{x}^c(y, \mathbf{w}), \lambda^c(y, \mathbf{w}), \mathbf{w}, y) \equiv 0$

Effects of input prices w and output level *y*

- FONC identity
- (mathematical tools: implicit function theorem or chain rule)
- $\begin{bmatrix} \mathbf{x}^c_{(\mathbf{w},y)} \\ \lambda^c_{(\mathbf{w},y)} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{x}^c_{\mathbf{w}} & \mathbf{x}^c_y \\ \lambda^c_{\mathbf{w}} & \lambda^c_y \end{bmatrix} = -\mathbf{H}^{-1}$, symmetric

  $\Rightarrow$ $\mathbf{x}^c_{\mathbf{w}}$ and λ^c_y symmetric, and $\mathbf{x}^c_y = (\lambda^c_{\mathbf{w}})^T$

More properties of $\mathbf{x}^c(y, \mathbf{w})$…

- Constraint restrictions: $\frac{\partial g}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}^c}{\partial \mathbf{w}} = 0, \quad \frac{\partial g}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{x}^c}{\partial y} = 1$
- Sign restrictions:

$$\mathbf{x}^c_{\mathbf{w}} = \frac{\partial \mathbf{x}^c}{\partial \mathbf{w}}, \text{ negative semi-definite}$$

- $\partial x_i^c / \partial w_i \leq 0$ for all i, downward sloping input demand function

Homogeneity Property

- Input demand function $\mathbf{x}^c(y, \mathbf{w})$
  - Homogeneous of degree zero in prices $\mathbf{w}$
- Implications
  - For each input demand, sum of all price elasticities must be equal to zero

Two inputs, *n*=2

- All results obtained earlier apply
- Additional implications
  - Price response is larger (smaller) when the isoquant is "flatter" ("more curved"), Figure 7.2
  - Cross price effect is always positive (how does it compare to the profit maximizing case?)

General case: *m* outputs, *n* inputs

- All results obtained earlier apply (not the *n*=2 specific results)
  - Input demand is homogeneous of degree zero in prices $\mathbf{w}$
  - Input demand functions tend to be downward sloping
  - Symmetry restrictions hold
  - Output effects, $\partial \mathbf{x}^c / \partial \mathbf{y}$ can be positive or negative, depends on the nature of technology