Question 1:

HW6 Answer Key

$u(x_1, x_2) = \ln x_1 + x_2$

utility function is quasiconcave and quasi-convex since $f(x) = \ln x_1$ is concave & $g(x) = x_2$ is both concave & convex.

$\Rightarrow$ FOC's are both necessary & sufficient for an interior solution.

Part A:

$$L = \ln x_1 + x_2 + \lambda[y - p_1 x_1 - p_2 x_2]$$

$$\left.\begin{aligned} L_1 &= \frac{1}{x_1} - \lambda p_1 = 0 \\ L_2 &= 1 - \lambda p_2 = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \frac{1}{x_1} &= \lambda p_1 \quad (1) \\ 1 &= \lambda p_2 \quad (2) \end{aligned}$$

$$L_\lambda = y - p_1 x_1 - p_2 x_2 = 0 \Rightarrow y = p_1 x_1 + p_2 x_2 \quad (3)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{1}{x_1} = \frac{p_1}{p_2}$$

$$\boxed{x_1^M = \frac{p_2}{p_1} \quad (4)}$$

substituting (4) into (3)

$$y = p_1\left(\frac{p_2}{p_1}\right) + p_2 x_2$$

$$y = p_2 + p_2 x_2$$

$$= p_2(1 + x_2)$$

$$\boxed{\frac{y}{p_2} - 1 = x_2^M}$$

$$\boxed{V(p, m) = \ln\left(\frac{p_2}{p_1}\right) + \frac{y}{p_2} - 1}$$

Part B:

Using duality

$$V(p,m) = \ln\left(\frac{p_2}{p_1}\right) + \frac{y}{p_2} - 1$$

$$u = \ln\left(\frac{p_2}{p_1}\right) + \frac{e(p,u)}{p_2} - 1$$

$$u - \ln\left(\frac{p_2}{p_1}\right) + 1 = \frac{e(p,u)}{p_2}$$

$$\boxed{u p_2 - p_2 \ln\left(\frac{p_2}{p_1}\right) + p_2 = e(p,u)} = p_2(u+1) + - p_2 \ln\left(\frac{p_2}{p_1}\right)$$

Using derivatives of $e(p,u)$ w/ respect to p to get the hicksian demand functions.

$$\frac{\partial e(p,u)}{\partial p_1} = h_1(p,u)$$

First, rearrange the $e(p,u)$.

$$u \cdot p_2 - p_2 \ln p_2 + p_2 \ln p_1 + p_2 = e(p,u)$$

$$\frac{\partial e(p,u)}{\partial p_1} = \boxed{\frac{p_2}{p_1} = h_1(p,u)}$$

$$\frac{\partial e(p,u)}{\partial p_2} = u - \ln p_2 - 1 + \ln p_1 + 1$$

$$= u + \ln p_1 - \ln p_2$$

$$= \boxed{u + \ln\left(\frac{p_2}{p_1}\right) = h_2(p,u)}$$

Part C:

$$\frac{\partial h_2(p,u)}{\partial p_2} = \frac{\partial f_2(p,y)}{\partial p_2} + \frac{\partial f_2(p,y)}{\partial m} \cdot f_2(p,y)$$

$$\frac{\partial h_2(p,u)}{\partial p_2} = \frac{-1}{p_2}$$

$$\frac{\partial f_2(p,m)}{\partial p_2} = -\frac{y}{p_2^2}$$

$$\frac{\partial f_2(p,m)}{\partial m} = \frac{1}{p_2}$$

$$\frac{\partial f_2(p,y)}{\partial p_2} + \frac{\partial f_2(p,y)}{\partial m} \cdot f_2(p,y) = \frac{-y}{p_2^2} + \frac{1}{p_2} \cdot \left(\frac{y}{p_2} - 1\right)$$

$$= \frac{-y}{p_2^2} + \frac{y}{p_2^2} - \frac{1}{p_2}$$

$$= -\frac{1}{p_2}$$

$$= \frac{\partial h_2(p,u)}{\partial p_2}$$

Part D)

$$\frac{\partial x_2(y,p)}{\partial p_1} < \frac{\partial x_1(y,p)}{\partial p_2} \quad **$$

We also know that $\frac{\partial x_1^c}{\partial p_2} = \frac{\partial x_2^c}{\partial p_1}$ ① by symmetry.

From the slutzky equation, we know that

$$\frac{\partial x_1^c}{\partial p_2} = \frac{\partial x_1(y,p)}{\partial p_2} + \frac{\partial x_1(y,p)}{\partial y} \cdot x_2(y,p) \quad ②$$

Combining equations ① & ②

$$\frac{\partial x_1(y,p)}{\partial p_2} + \frac{\partial x_1(y,p)}{\partial y} \cdot x_2(y,p) = \frac{\partial x_2(y,p)}{\partial p_1} + \frac{\partial x_2(y,p)}{\partial y} \cdot x_1(y,p)$$

Knowing that $\frac{\partial x_2(y,p)}{\partial p_1} < \frac{\partial x_1(y,p)}{\partial p_2}$

This implies that

$$\frac{\partial x_1}{\partial y} \cdot x_2(y,p) < \frac{\partial x_2}{\partial y} \cdot x_1(y,p)$$

Multiplying both sides by $\frac{y}{x_1(y,p)\, x_2(y,p)}$

$$\frac{\partial x_1}{\partial y} \cdot \frac{y}{x_1(y,p)} < \frac{\partial x_2}{\partial y} \frac{y}{x_2(y,p)}$$

$$\varepsilon_{Y1} < \varepsilon_{Y2}$$

From Part A, we know that

$$X_1 = \frac{P_2}{P_1}$$

$$X_2 = \frac{Y}{P_2} - 1$$

$$\left.\begin{array}{l} \frac{\partial X_1}{\partial Y} = 0 \\ \\ \frac{\partial X_2}{\partial Y} = \frac{1}{P_2} \end{array}\right\} \Rightarrow \varepsilon_{Y1} = 0 \quad , \quad \varepsilon_{Y2} = \frac{1}{P_2} \cdot \frac{Y}{\left(\frac{Y}{P_2} - 1\right)}$$

NOTE: $X_2 > 0$ since we are assuming an interior solution.

$\Rightarrow \varepsilon_{Y1} < \varepsilon_{Y2}$ since $\varepsilon_{Y2} > 0$.

Part e). and f)

CV - or compensating variation, is the adjustment in income that returns the consumer to the original utility after an economic change has occurred

In the case of a positive economic change (such as a fall in price of a good). CV is often referred to as the maximum a consumer is willing to pay in order to have the economic change happen. When there is a negative economic change, CV is the minimum the consumer needs in order to accept the economic change.

EV - or equivalent variation, is the adjustment in income that changes the consumer's utility equal to the level that would occur if the event had happened

In the case of a positive economic change, such as a fall in price, EV would be the increase in income that would given the consumer the same additional utility that would happen if a price did fall. In the case of a negative economic change, EV would be the amount of income that would be taken away to lower the consumer's utility to the level that would happen if the change had happened. (for avoiding the change).

Here in question e), the consumers have the "right" to stay at the status quo. Thus CV should be used. It measures the minimum the consumer needs in order to accept the price increase.

In question f), although the consumer wants to stay at the status quo, the bill will force them into the status after change. In other words, the consumer only has "right" to be in the state after change. Thus EV should be used. It measures the amount of income that the consumer is willing to pay to avoid the price increase.

Part E.

Compensating Variation

- the additional money an individual would need in order to reach its initial utility AFTER a change in prices.

$$C.V. = e(P_1, V_0) - e(p_0, V_0)$$

$$= 202.773 - 200$$

$$= 2.773$$

NOTE E.V. = C.V, also for part f)

Part F.

Equivalent Variation

- the amount of additional \$ a consumer would be WTP BEFORE a price increase in order to prevent the change.

$$E.V. = e(p_1, V_1) - e(p_0, V_1) \quad \text{s.t. } V_1 = V(p_1, y_1) = u^*(p_1, y_1)$$

and $p_1 = (4, 2)$

$y = 200$

$$= 200 - 197.23$$

$$\approx 2.773$$

See the next page for the mathematical calculations

Part E & Part F Explanation

$$V(p_0, M_0) = \ln\left(\frac{2}{1}\right) + \frac{200}{2} - 1$$

$$= \ln 2 + 100 - 1$$

$$= \ln 2 + 99$$

$$V(p_1, M') = \ln\left(\frac{2}{4}\right) + \frac{200}{2} - 1$$

$$= \ln\left(\tfrac{1}{2}\right) + 99$$

$$= 99 - \ln 2$$

$$e(p_0, V_0) = 200$$

$$e(p_1, V_1) = 200$$

$$e(p_1, V_0) = (\ln 2 + 99)2 - 2\ln\left(\tfrac{1}{2}\right) + 2$$

$$= 2\ln 2 + 2 \cdot 99 + 2\ln 2 + 2$$

$$\approx 202.773$$

$$e(p_0, V_1) = (99 - \ln 2) \cdot 2 - 2\ln 2 + 2$$

$$= 2 \cdot 99 - 2\ln 2 - 2\ln 2 + 2$$

$$\approx 197.227$$

Background Work

(Necessary if you want partial credit even w/ math errors)

$$CV = e(p_1, V_0) - e(p_0, V_0)$$

$$\approx 202.773 - 200$$

$$\approx 2.773$$

$$\underline{CV = EV}$$

$$EV \approx e(p_1, V_0) - e(p_0, V_1)$$

$$\approx 200 - 197.227$$

$$\approx 2.773$$

Part G.

For quasilinear utility functions

$$u(x_1, x_2, \ldots x_n) = x_1 + \eta(x_{-i}) \quad \text{s.t. } x_{-i} = \{x_2, x_3 \ldots x_n\}$$

w/o loss of generality, we say that good 1 doesn't enter η.
There are two ways that you could answer this problem.

Method 1:
Show that the Marshallian and Hicksian demands coincide for good x_1 (or in general the good that doesn't enter the utility fnctn linearly).

here $f_1(m, p) = \frac{p_2}{p_1}$ for this specific function.

$$h_1(p, u) = \frac{p_2}{p_1}$$

graphically, you can then show that

$$h_1(u', p) = h_1(u^0, p)$$

and thus $CV = EV$

This method is <u>NOT</u> the best method, but will be accepted.

Method 2:

<u>UMP</u> $$\max_x \eta(x_2, x_3, \ldots x_n) + x_1 + \lambda\left[m - x_1 - \sum_{i=2}^{n} p_i x_i\right]$$

w/o loss of generalization, we can normalize the prices so that $p_1 = 1$.

Assume that $\eta(x_{-i})$ is quasiconcave so that we can gaurantee an interior solution.

$$L_1: \quad 1 - \lambda = 0 \quad \Rightarrow \lambda = 1$$

$$L_i: \quad \frac{\partial \eta}{\partial x_i} - \lambda p_i = 0 \quad \forall i = 2, 3, \ldots n$$

$\therefore \dfrac{\partial \eta}{\partial x_i} = p_i \quad \forall\ i = 2, 3, \ldots, n$

$\Rightarrow$ we have $n-1$ unknowns & $n-1$ equations.

The system can be solved to find all $g_i(p_{-1})$ $\forall\ i = 2, \ldots n$

NOTE p_{-1} means that the ~~prices~~ marshallian demands of goods $2 \ldots n$ only depend on prices $2 \ldots n$.

$$u^*(p, M) = \eta(g_{-1}(p_{-1})) + \underbrace{m - p_{-1}\, g_{-1}(p_{-1})}_{\text{Marshallian demand of good 1 due to Walras's Law}}$$

$$= \underbrace{\eta(g_{-1}(p_{-1})) - p_{-1}(g_{-1}(p_{-1}))}_{\tilde{v}(p_{-1})} + m$$

$$= \tilde{v}(p_{-1}) + m$$

$$v(p, m) = \tilde{v}(p_{-1}) + m$$

By duality: $e(p, u) = u - \tilde{v}(p_{-1})$

EV & CV

$$EV = e(p^0, u^1) - m^0 = \tilde{v}(p^1_{-1}) + m^0 - \tilde{v}(p^0_{-1}) + m^0$$

$$= \tilde{v}(p^1_{-1}) - \tilde{v}(p^1_0)$$

$$CV = -e(p^1, u^0) + m^0 = m^0 - \tilde{v}(p^0_{-1}) + m^0 - \tilde{v}(p^1_{-1}) =$$

$$= \tilde{v}(p^1_{-1}) - \tilde{v}(p^0_{-1})$$

$CV = EV$ QED

Question 2:

$$u(x_1, x_2) = (x_1^{1/2} + x_2^{1/2})^2$$

The benefit function is defined as

$$B(x, u) = \max_{\beta} \{\beta : u(x - g\beta), x - g\beta \geq 0\}$$

Part A:

The function is maximized when:

$$u(x - g\beta) = u$$

$$u = ((x_1 - \beta)^{1/2} + x_2^{1/2})^2$$

Solving for β:

$$u^{1/2} = (x_1 - \beta)^{1/2} + x_2^{1/2}$$

$$u^{1/2} - x_2^{1/2} = (x_1 - \beta)^{1/2}$$

$$(u^{1/2} - x_2^{1/2})^2 = x_1 - \beta$$

$$\beta = x_1 + -(u^{1/2} - x_2^{1/2})^2$$

$$\Rightarrow B(x, u) = x_1 - (u^{1/2} + - x_2^{1/2})^2$$

Part B:

We need to show that the following duality relationship holds:

$$B(x,u) = \min_{p} \left\{ p^T x - E(p,u) : p^T g = 1,\ p \geq 0 \right\}$$

We are given that $E(p,u) = \dfrac{u p_1 p_2}{(p_1 + p_2)}$

We also know that $p^T g = 1$

since $g = (1,0)$ $\quad p^T g = p_1(1) + p_2(0) = 1$

$$\Rightarrow p_1 = 1$$

Substituting this into the objective function along w/ $E(p,u)$

$$\min_{p} \left\{ x_1 + p_2 x_2 - \frac{u p_2}{(1+p_2)} \right\}$$

$$\text{FOC: } \frac{\partial B}{\partial p_2} = x_2 - \left(\frac{u(1+p_2) - u p_2}{(1+p_2)^2} \right)$$

$$= x_2 - \left(\frac{u + u p_2 - u p_2}{(1+p_2)^2} \right)$$

$$= x_2 - \frac{u}{(1+p_2)^2}$$

$$\frac{\partial B}{\partial p_2} = 0 \quad \Rightarrow \quad x_2 = \frac{u}{(1+p_2)^2}$$

since $p g = 1$, cce holds

In the unconstrained minimization problem, Let $L = x_1 + p_2 x_2 - \dfrac{u p_2}{1+p_2}$

$$\frac{\partial L}{\partial p_2} = x_2 - \frac{(1+p_2)u - u p_2}{(1+p_2)^2}$$

$$= x_2 - \frac{1}{(1+p_2)^2}$$

$$\frac{\partial^2 L}{\partial p_2^2} = \frac{2(1+p_2)}{(1+p_2)^4} = \frac{2}{(1+p_2)^3} > 0$$

L is a concave function. So x_2^* is a global minimum.

$$x_2(1+p_2)^2 = u$$

$$\frac{u}{x_2} = (1+p_2)^2$$

$$1+p_2 = \left(\frac{u}{x_2}\right)^{1/2}$$

$$p_2 = \left(\frac{u}{x_2}\right)^{1/2} - 1$$

Substituting back into the objective function

$$x_1 + \left[\left(\frac{u}{x_2}\right)^{1/2} - 1\right]x_2 - \frac{u\left[\left(\frac{u}{x_2}\right)^{1/2} - 1\right]}{\left[1 + \left(\frac{u}{x_2}\right)^{1/2} - 1\right]}$$

$$= x_1 + u^{1/2}x_2^{1/2} - \frac{u\left[\left(\frac{u}{x_2}\right)^{1/2} - 1\right]}{\left(\frac{u}{x_2}\right)^{1/2}} + -x_2$$

$$= x_1 + u^{1/2}x_2^{1/2} + u + u\left(\frac{x_2}{u}\right)^{1/2} + -x_2$$

$$= x_1 + u^{1/2}x_2^{1/2} - u + u^{1/2}x_2^{1/2} + -x_2$$

$$= x_1 + -x_2 + 2u^{1/2}x_2^{1/2} + u$$

$$= x_1 - (-u - 2u^{1/2}x_2^{1/2} + x_2)$$

$$= x_1 - (u^{1/2} - x_2^{1/2})^2$$

$$= B(u,x)$$