Question 1 : Part A

Cost functions are HOD zero in w.

$$x_1^c(\lambda w) = 1 + 3\lambda^{-1/2} w_1^{-1/2} \lambda^a w_2^a$$

$$= 1 + \lambda^{-1/2+a} \left(3 w_1^{-1/2} w_2^a\right)$$

$$\Rightarrow a = 1/2$$

$$x_2^c(\lambda w) = 1 + b \lambda^{1/2} w_1^{1/2} \lambda^c w_2^c$$

$$= 1 + b\, \lambda^{1/2+c} w_1^{1/2} w_2^c$$

$$\Rightarrow \cancel{b}\, c = -1/2$$

$$\frac{\partial x_1^c}{\partial w_2} = \frac{\partial x_2^c}{\partial w_1}$$

$$\frac{\partial x_1^c}{\partial w_2} = \frac{3}{2} w_1^{-1/2} w_2^{-1/2}$$

$$\frac{\partial x_2^c}{\partial w_1} = \frac{b}{2} w_1^{-1/2} w_2^{-1/2}$$

$$\Rightarrow b = 3$$

$$\boxed{a = 1/2 \ , \quad b = 3 \ , \quad c = -1/2}$$

Question 1 Part B

Superadditivity of cost functions means $c(w^1+w^2, y) \geq c(w^1, y) + c(w^2, y)$.
with w^1 and w^2 be arbitrarily chosen factor prices and y be any output level.
Note w^1 and w^2 are both vectors of input prices.

Cost functions are HOD 1 in w and concave in w. Let $\lambda = 1/2$.
Concavity in w implies the following:

$$c\left(\frac{(w^1+w^2)}{2}, y\right) \geq \frac{c(w^1,y)}{2} + \frac{c(w^2,y)}{2} \qquad \text{(A)}$$

Homogeneity of degree 1 in w implies

$$c\left(\frac{w^1+w^2}{2}, y\right) = \frac{1}{2} c((w^1+w^2), y) \qquad \text{(B)}$$

Substituting (B) into (A) we get the following:

$$\frac{1}{2} c(w^1+w^2, y) \geq \frac{c(w^1,y)}{2} + \frac{c(w^2,y)}{2}$$

$$\therefore c(w^1+w^2, y) \geq c(w^1,y) + c(w^2,y)$$

To show that this implies that c is non-decreasing in w, consider a price change of $\varepsilon > 0$ in factor 1, w/o loss of generality.

$$c(w_1+\varepsilon, w_2, \ldots, w_n, y) \geq c(w_1, w_2, \ldots w_n, y) + c(\varepsilon, 0, \ldots, 0, y)$$

$$\geq c(w_1, w_2, \ldots, w_n, y)$$

Where the last inequality comes from the fact that cost functions are non-negative.

$\Rightarrow c$ is non-decreasing in w.

Question 1 Part C:

True. It is an imposter!
cost functions are HOD 1 in w. This function is HOD 1/2 in prices.

$$C(\lambda w_1, \lambda w_2, y) = (\sqrt{\lambda w_1} + \sqrt{\lambda w_2})y$$
$$= \lambda^{1/2}(\sqrt{w_1} + \sqrt{w_2})y$$

Question 1 Part D:

$$y = \sum_{i=1}^{n} \alpha_i x_i$$

The inputs in this technology are perfect substitutes, so cost minimization implies that the inputs with the lowest input price will be used.

Thus $y = \alpha x_i \Rightarrow x_i = \frac{y}{\alpha_i}$

Thus $C(w, y) = \min\{w_i x_i\} = \min\{\frac{w_i}{\alpha_i} y\}$

Question 2 : Part A

$$\min_{x_1, x_2, y} \; w_1 x_1 + w_2 x_2 \quad \text{s.t.} \quad y = (x_1^{1/2} + x_2^{1/2})^2$$

NOTE: The cost function $w_1 x_1 + w_2 x_2 = C(w, x)$ is convex in x.

$$h(x) = y - f(x_1, x_2)$$
$$= y - (x_1^{1/2} + x_2^{1/2})^2 \text{ is quasiconvex since}$$

$f(x_1, x_2) = (x_1^{1/2} + x_2^{1/2})^2$ is quasiconcave. We know this b/c of the following.

Let $g(x_1, x_2) = x_1^{1/2} + x_2^{1/2}$

Then, for any $x, z \in \mathbb{R}^n_{++}$ we have the following:

Let $\lambda \in (0,1)$, then

$$\lambda g(z) + (1-\lambda) g(x) = \lambda z_1^{1/2} + \lambda z_2^{1/2} + (1-\lambda) x_1^{1/2} + (1-\lambda) x_2^{1/2}$$
$$= \sum_{i=1}^{2} \lambda z_i^{1/2} + (1-\lambda) x_i^{1/2}$$
$$< \sum_{i=1}^{2} (\lambda z_i + (1-\lambda) x_i)^{1/2}$$
$$= g(\lambda z + (1-\lambda) x)$$

This implies that $g(\cdot)$ is strictly concave.

Since f is a monotone increasing transformation of g, it follows that f is strictly quasi-concave.

Since $\frac{\partial h}{\partial x_1} \neq 0$ & $\frac{\partial h}{\partial x_2} \neq 0$, the C.Q. is satisfied.

$\Rightarrow$ our FOC's will be sufficient for an interior minimum.

At the optimum, we know that the marginal technical rate of substitution equals the wage share.

$$f_1 = \frac{1}{2} x_1^{-1/2} (x_1^{1/2} + x_2^{1/2})$$

$$f_2 = \frac{1}{2} x_2^{-1/2} (x_1^{1/2} + x_2^{1/2})$$

$$\frac{f_1}{f_2} = \frac{w_1}{w_2} \Rightarrow \frac{\frac{1}{2} x_1^{-1/2} \cancel{(x_1^{1/2} + x_2^{1/2})}}{\frac{1}{2} x_2^{-1/2} \cancel{(x_1^{1/2} + x_2^{1/2})}} = \frac{w_1}{w_2}$$

$$\frac{x_2^{1/2}}{x_1^{1/2}} = \frac{w_1}{w_2}$$

$$\left(\frac{x_2}{x_1}\right)^{1/2} = \frac{w_1}{w_2}$$

$$\frac{x_2}{x_1} = \left(\frac{w_1}{w_2}\right)^2$$

$$x_2 = x_1 \left(\frac{w_1}{w_2}\right)^2$$

$$y = \left(x_1^{1/2} + \left[x_1 \left(\frac{w_1}{w_2}\right)^2 \right]^{1/2} \right)^2$$

$$y = \left(x_1^{1/2} + x_1^{1/2} \left(\frac{w_1}{w_2}\right) \right)^2$$

$$y = \left(x_1^{1/2} \left(1 + \frac{w_1}{w_2} \right) \right)^2$$

$$y = x_1 \left(1 + \frac{w_1}{w_2} \right)^2$$

$$x_1^c = \frac{y}{\left(1 + \frac{w_1}{w_2}\right)^2} = \frac{y}{\left(\frac{w_2 + w_1}{w_2}\right)^2} = y \left(\frac{w_2}{w_1 + w_2}\right)^2$$

$$x_2^c = y\left(\frac{w_1}{w_1+w_2}\right)^2$$ by a similar calculation

$$c(w_1, w_2, y) = w_1 y\left(\frac{w_2}{w_1+w_2}\right)^2 + w_2 y\left(\frac{w_1}{w_1+w_2}\right)^2$$

$$= y\left(\frac{w_1 w_2^2 + w_2 w_1^2}{(w_1+w_2)^2}\right)$$

$$= y\left(\frac{w_1 w_2 \cancel{(w_1+w_2)}}{(w_1+w_2)^{\cancel{2}}}\right)$$

$$= y\left(\frac{w_1 w_2}{(w_1+w_2)}\right)$$

Question 2 Part B

Elasticity of substitution $\sigma = -\dfrac{d\ln\left(\frac{x_1}{x_2}\right)}{d\ln\left(\frac{f_1(x_1,x_2)}{f_2(x_1,x_2)}\right)}$

$$f_1 = x_1^{-1/2}(x_1^{1/2} + x_2^{1/2})$$
$$f_2 = x_2^{-1/2}(x_1^{1/2} + x_2^{1/2})$$

$$\frac{f_1}{f_2} = \frac{x_2^{1/2}}{x_1^{1/2}}$$

$$\ln\left(\frac{f_1}{f_2}\right) = \ln\left(\frac{x_2^{1/2}}{x_1^{1/2}}\right)$$

$$= \frac{1}{2}\ln\left(\frac{x_2}{x_1}\right)$$

$$E.S. = \frac{-d\ln\left(\frac{x_1}{x_2}\right)}{d\ln\left(\frac{f_1(x_1,x_2)}{f_2(x_1,x_2)}\right)}$$

$$= \frac{d\ln\left(\frac{x_2}{x_1}\right)}{d\ln\left(\frac{f_1(x_1,x_2)}{f_2(x_1,x_2)}\right)} = \frac{d\ln\left(\frac{x_2}{x_1}\right)}{\frac{1}{2}d\ln\left(\frac{x_2}{x_1}\right)} = \frac{1}{1/2} = 2$$

$\Rightarrow$ Elasticity of Substitution $=2$

Allen Elasticity of Substitution

$$\sigma_{ij} = \frac{\partial x_i^c}{\partial w_j} \cdot \frac{C}{x_i^c x_j^c}$$

$$\sigma_{ij} = \sigma_{ji} \quad \text{so } \sigma_{12} = \frac{\partial x_1^c}{\partial w_2} \cdot \frac{C}{x_1^c x_2^c}$$

$$= \frac{\partial x_2^c}{\partial w_1} \cdot \frac{C}{x_1^c x_2^c} = \sigma_{21}$$

$$x_1^c = y\left(\frac{w_2}{w_1+w_2}\right)^2$$

$$x_2^c = y\left(\frac{w_1}{w_1+w_2}\right)^2$$

from part A

$$\frac{\partial x_1^c}{\partial w_2} = \frac{2w_1w_2}{(w_1+w_2)^3}y$$

$$\sigma_{12} = \frac{2w_1w_2y}{(w_1+w_2)^3} \cdot \frac{y\left[\frac{w_1w_2}{w_1+w_2}\right]}{y^2\left[\frac{w_1^2w_2^2}{(w_1+w_2)^4}\right]}$$

$$= 2\frac{w_1w_2}{(w_1+w_2)^3} \cdot \frac{w_1w_2}{w_1+w_2} \cdot \frac{(w_1+w_2)^4}{(w_1w_2)^2}$$

$$= 2 \frac{(w_1 w_2)^2}{(w_1 + w_2)^4} \cdot \frac{(w_1 + w_2)^4}{(w_1 w_2)^2}$$

$$= 2$$

$$\Rightarrow AES = 2$$

Question 2 Part C: Shepard's Lemma

$$\frac{\partial C}{\partial w_1} = \frac{d}{dw_1}\left[y \frac{w_1 w_2}{(w_1 + w_2)}\right]$$

$$= y\left[\frac{w_2(w_1 + w_2) - w_1 w_2}{(w_1 + w_2)^2}\right]$$

$$= y\left[\frac{\cancel{w_2 w_1} + w_2^2 - \cancel{w_1 w_2}}{(w_1 + w_2)^2}\right]$$

$$= y\left[\frac{w_2^2}{(w_1 + w_2)^2}\right]$$

$$= x_1^c$$

$$\frac{\partial C}{\partial w_2} = \frac{d}{dw_2}\left[y \frac{w_1 w_2}{(w_1 + w_2)}\right]$$

$$= y\left[\frac{w_1(w_1 + w_2) - w_1 w_2}{(w_1 + w_2)^2}\right]$$

$$= y\left[\frac{w_1^2 + w_1 w_2 - w_1 w_2}{(w_1 + w_2)^2}\right]$$

$$= y\left[\frac{w_1^2}{(w_1 + w_2)^2}\right]$$

$$= x_2^c$$

Thus Shepard's Lemma has been verified.

Question 2 : Part D

$$\text{Ratio of factor shares} = \frac{w_1 x_1^c(w_1, w_2, y)}{w_2 x_2^c(w_1, w_2, y)}$$

$$= \frac{w_1 \left(\frac{w_2^2}{(w_1 + w_2)^2} \right)}{w_2 \left(\frac{w_1^2}{(w_1 + w_2)^2} \right)}$$

$$= \frac{w_1 w_2^2}{(w_1 + w_2)^2} \cdot \frac{(w_1 + w_2)^2}{w_1^2 w_2}$$

$$= \frac{w_1 w_2^2}{w_1^2 w_2} = \frac{w_2}{w_1}$$

$$= \left(\frac{w_1}{w_2} \right)^{-1}$$

Question 3:

$$C(w,y) = \min_{x_1, x_2} \; w_1 x_1 + w_2 x_2 \quad \text{s.t.} \quad y = x_1^{\alpha} x_2^{1-\alpha} \quad , \alpha < 1$$

$y = x_1^{\alpha} x_2^{1-\alpha}$ is a Cobb-Douglas production function.
Thus, we know that it is quasi-concave
(A more formal proof similar to question 2 can be formulated.)

$h(x_1, x_2) = y - x_1^{\alpha} x_2^{\alpha-1}$ is quasiconvex

$\nabla h(x, x_2) = [-\alpha x_1^{\alpha-1} x_2^{1-\alpha}, (1-\alpha) x_1^{\alpha} x_2^{\alpha-2}] \neq 0$ so the C.Q. condition holds.

$\Rightarrow$ our solution to the minimization problem will be an interior minimum.

AT the optimum: MRTS = price ratio

$$MRTS = \frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{(1-\alpha) x_1^{\alpha} x_2^{\alpha-2}} = \frac{w_1}{w_2}$$

$$\frac{\alpha}{1-\alpha} \frac{x_2}{x_1} = \frac{w_1}{w_2}$$

$$x_2 = \frac{(1-\alpha)}{\alpha} \frac{w_1}{w_2} x_1$$

Plugging x_2 into the production function, you get

$$x_1^{\alpha} \left(\frac{1-\alpha}{\alpha} \frac{w_1}{w_2} x_2 \right)^{1-\alpha} = y$$

$$\boxed{x_1(w,y) = y \left(\frac{\alpha}{1-\alpha} \frac{w_2}{w_1} \right)^{1-\alpha}}$$

$$x_1 = \frac{\alpha}{(1-\alpha)} \cdot \frac{w_2}{w_1} \cdot x_2$$

$$y = \left(\frac{\alpha}{1-\alpha} \frac{w_2}{w_1} x_2 \right)^{\alpha} x_2^{1-\alpha}$$

$$\boxed{x_2(w,y) = y \left(\frac{1-\alpha}{\alpha} \cdot \frac{w_1}{w_2} \right)^{\alpha}}$$

The cost function is:

$$\boxed{C(w,y) = w_1 y \left(\frac{\alpha}{1-\alpha} \cdot \frac{w_2}{w_1}\right)^{1-\alpha} + w_2 y \left(\frac{1-\alpha}{\alpha} \frac{w_1}{w_2}\right)^{\alpha}}$$

(i) Show that the $MC = LAC$

$$LAC = \frac{C(w,y)}{y}$$

$$= w_1 \left(\frac{\alpha}{1-\alpha} \cdot \frac{w_2}{w_1}\right)^{1-\alpha} + w_2 \left(\frac{1-\alpha}{\alpha} \frac{w_1}{w_2}\right)^{\alpha}$$

$$MC = \frac{\partial C(w,y)}{\partial y}$$

$$= w_1 \left(\frac{\alpha}{1-\alpha} \cdot \frac{w_2}{w_1}\right)^{1-\alpha} + w_2 \left(\frac{1-\alpha}{\alpha} \cdot \frac{w_1}{w_2}\right)^{\alpha}$$

Assume that x_2 is fixed ($x_2 = \bar{x}_2$) in the short-run.

$$SC(w,y) = \min_{x_1} \quad x_1 w_1 + \bar{x}_2 w_2 \quad \text{s.t.} \quad y = x_1^{\alpha} \bar{x}_2^{1-\alpha} \quad \text{s.t.} \ \alpha < 1$$

$$h(x) = y - x_1^{\alpha} \bar{x}_2^{1-\alpha}$$
$$h_1(x) = -\alpha x_1^{\alpha-1} \bar{x}_2^{1-\alpha} < 0$$

$\Rightarrow$ $h(x)$ is convex and the C.Q. condition holds
our solution to the constrained minimization problem will be an interior solution.

$$y = x_1^{\alpha} \bar{x}_2^{1-\alpha}$$

$$x_1 = \left(\frac{y}{\bar{x}_2^{1-\alpha}}\right)^{1/\alpha}$$

$$\boxed{x_1 = y^{1/\alpha} \bar{x}_2^{\left(\frac{\alpha-1}{\alpha}\right)}}$$

$$\boxed{SC = y^{1/\alpha} \bar{x}_2^{\left(\frac{\alpha-1}{\alpha}\right)} w_1 + \bar{x}_2 w_2}$$

ii) Show that for every level of the fixed input, the SAC and the LAC are equal at the minimum of the SAC.

$$SAC(w,y) = \frac{SC(w,y)}{y} = w_1 y^{\left(\frac{1-\alpha}{\alpha}\right)} \bar{x}_2^{\left(\frac{\alpha-1}{\alpha}\right)} + \frac{w_2 \bar{x}_2}{y}$$

The minimum SAC is found by taking the F.O.C.

$$\frac{\partial SAC}{\partial y} = \left(\frac{1-\alpha}{\alpha}\right) w_1 y^{\frac{1-2\alpha}{\alpha}} \bar{x}_2^{\frac{\alpha-1}{\alpha}} - \frac{w_2 \bar{x}_2}{y^2} = 0$$

Solving for $y^* = \left[\frac{\alpha}{1-\alpha} \; \frac{w_2}{w_1}\right]^{\alpha} \bar{x}_2$

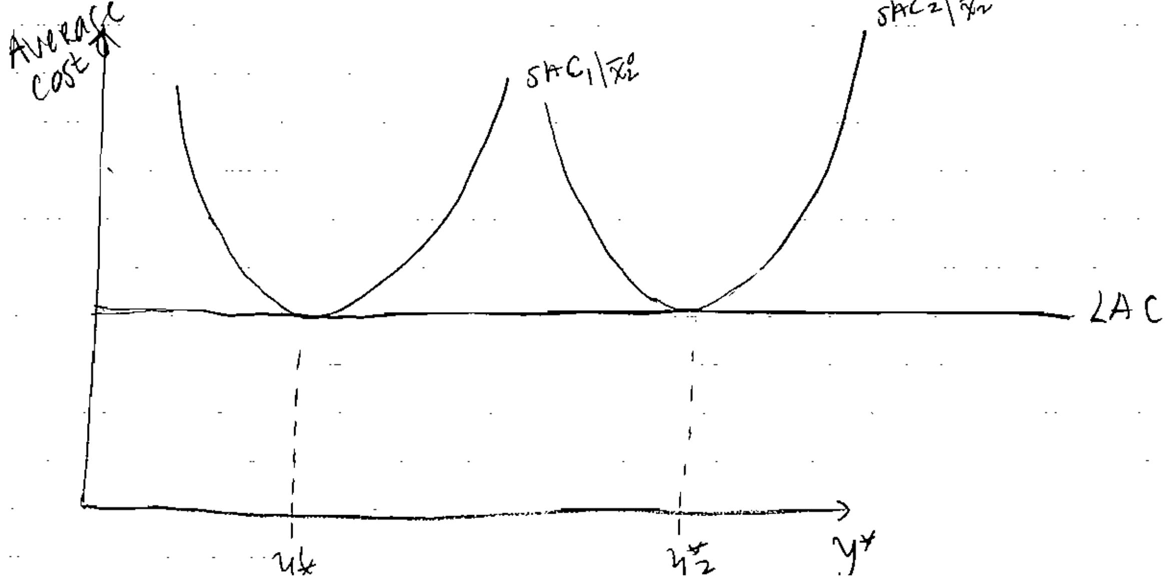
$$\text{SOSC}: \quad \frac{\partial^2 SAC}{\partial y^2} = \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1-2\alpha}{\alpha}\right) w_1 y^{\frac{1-3\alpha}{\alpha}} \bar{x}_2^{\frac{\alpha-1}{\alpha}} - \frac{2 w_2 \bar{x}_2}{y^3} > 0$$

Plugging y^* into SAC gives

$$SAC(y^*, w) = w_1 \left(\frac{\alpha}{1-\alpha} \; \frac{w_2}{w_1}\right)^{1-\alpha} + w_2 \left(\frac{1-\alpha}{\alpha} \; \frac{w_1}{w_2}\right)^{\alpha}$$

$$= LAC$$

(iii) Illustration of Results



Question 4.

A). setup the problem as

$$\min_{x_1, x_2, y} \frac{w_1x_1+w_2x_2}{y} \quad \text{s.t } y = g(x_1, x_2)$$

$$\mathcal{L} = \frac{w_1x_1+w_2x_2}{y} + \lambda(y - g(x_1, x_2))$$

After checking for CQ condition, and by taking the derivative of this with respect to x_1, x_2, y and λ, and solving the resulting FONC, we can arrive at the candidates $x_i(w_1, w_2)$ and $y(w_1, w_2)$. After checking for SOSC or the special case, we can finally get the optimal factor demands $x_i^*(w_1, w_2)$ and output supply $y^*(w_1, w_2)$. The value function is then:

$$AC^*(w_1, w_2) = \frac{w_1 x_1^*(w_1, w_2) + w_2 x_2^*(w_1, w_2)}{y^*(w_1, w_2)}$$

Suppose that both input prices are scaled by a factor k. The problem now is to solve:

$$\min_{x_1, x_2, y} \frac{kw_1x_1+kw_2x_2}{y} \quad \text{s.t. } y = g(x_1, x_2)$$

or
$$\min_{x_1, x_2, y} k \cdot \frac{w_1x_1+w_2x_2}{y} \quad \text{s.t } y = g(x_1, x_2)$$

which naturally yields the same factor demands and output supply $x_i^*(kw_1, kw_2) = x_i^*(w_1, w_2)$, $y^*(kw_1, kw_2) = y^*(w_1, w_2)$. This implies the input demands and output supplies are HOD zero

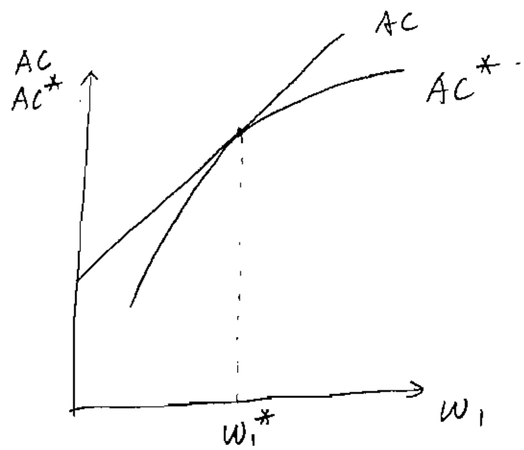
It follows that

$$AC^*(kw_1, kw_2) = \frac{kw_1 x_1^*(kw_1, kw_2) + kw_2 x_2^*(kw_1, kw_2)}{y^*(kw_1, kw_2)}$$

$$= k\,\frac{w_1 x_1^*(w_1, w_2) + w_2 x_2^*(w_1, w_2)}{y^*(w_1, w_2)}$$

$$= k\,AC^*(w_1, w_2)$$

B)



where $AC = \dfrac{w_1 x_1 + w_2 x_2}{f(x_1, x_2)}$

$AC^*(w_1, w_2, y)$

AC ~~itself~~ is linear in w_1. AC^* is the minimum average cost at the "correct" values of x_1, x_2 and y. Therefore for $w_1 \neq w_1^*$, $AC^* < AC$ while for $w_1 = w_1^*$, $AC^* = AC$. Since AC is a linear function of w_1, AC^* must be curved down at $w_1 \neq w_1^*$, AC^* must be concave in w_1.

c) From the envelope theorem

$$\frac{\partial AC^*(w_1, w_2)}{\partial w_1} = \frac{\partial L(x^*(w_1,w_2), y^*(w_1,w_2), \lambda^*(w_1,w_2), w_1, w_2)}{\partial w_1}$$

$$= \frac{x_1^*(w_1, w_2)}{y^*(w_1, w_2)}$$

The slope of the value function in w_1 is the input-output ratio for input 1

d. We have established that $\frac{\partial AC^*(w_1,w_2)}{\partial w_1}=\frac{x_1^*(w_1,w_2)}{y^*(w_1,w_2)}$, and that AC^* is concave in w_1 (i.e. that $\frac{\partial AC^*(w_1,w_2)}{\partial w_1}$ is a decreasing function of w_1). It follows that $\frac{x_1^*(w_1,w_2)}{y^*(w_1,w_2)}$ is a decreasing function of w_1.

e. Having established that $\frac{\partial(x_i^*/y^*)}{\partial w_i}\le 0$, it is trivial to note that:

$\frac{\partial(x_i^*/y^*)}{\partial w_i}=\frac{y^*(\partial x_i^*/\partial w_i)-x_i^*(\partial y^*/\partial w_i)}{y^{*2}}\le 0$, or $\frac{y^*(\partial x_i^*/\partial w_i)}{y^{*2}}\le\frac{x_i^*(\partial y^*/\partial w_i)}{y^{*2}}$. If we multiply both sides by $\frac{yw_i}{x_i}>0$, this yields: $\frac{w_i\partial x_i^*}{x_i\partial w_i}\le\frac{w_i\partial y^*}{y^*\partial w_i}$ or $\frac{d\ln x_i^*}{d\ln w_i}\le\frac{d\ln y^*}{d\ln w_i}$.

f. The Lagrangian to the primal-dual model is:

$$L=\frac{w_1x_1+w_2x_2}{y}-AC^*(w_1,w_2)+\lambda[y-g(x_1,x_2)],$$

from which the FONC and optimal input demands and output supply will be identical to those derived from the Lagrangian in a. Now, the primal dual results tell us that, because the input prices do not appear in the constraints (ie. h_α=0), that:

$$\begin{bmatrix}\frac{1}{y^*} & 0 & \frac{-x_1^*}{y^{*2}} & 0\\ 0 & \frac{1}{y^*} & \frac{-x_2^*}{y^{*2}} & 0\end{bmatrix}\begin{bmatrix}\frac{\partial x_1^*}{\partial w_1} & \frac{\partial x_1^*}{\partial w_2}\\ \frac{\partial x_2^*}{\partial w_1} & \frac{\partial x_2^*}{\partial w_2}\\ \frac{\partial y^*}{\partial w_1} & \frac{\partial y^*}{\partial w_2}\\ \frac{\partial \lambda^*}{\partial w_1} & \frac{\partial \lambda^*}{\partial w_2}\end{bmatrix}$$

is negative semi-definite (because this is a minimization problem). Thus the diagonal terms of the matrix produced must be non-positive. It follows that $\frac{(\partial x_i^*/\partial w_i)}{y^*}-\frac{x_i^*(\partial y^*/\partial w_i)}{y^{*2}}\le 0$, from where the proof proceeds exactly as in e.

g. The behavior predicted by this model exactly matches that which results if a profit maximizing firm were faced with a price exactly equal to its minimum average cost. This is all that is meant by the identity: $x_1^*(w_1,w_2)\equiv x_1^p(w_1,w_2,p^*(w_1,w_2))$, where $p^*(w_1,w_2)=AC^*(w_1,w_2)$. In other words, a competitive firm facing an input price that is just equal to its minimum average production cost, will demand exactly as much input 1 as it would have were it selecting inputs to minimize the average cost of production.

h. Differentiating the identity with respect to w_1, yields:

$$\frac{\partial x_1^*}{\partial w_1} \equiv \frac{\partial x_1^p}{\partial w_1} + \frac{\partial x_1^p}{\partial p}\frac{\partial p^*}{\partial w_1^*}$$ or:

♦ $$\frac{\partial x_1^*}{\partial w_1} \equiv \frac{\partial x_1^p}{\partial w_1} + \frac{\partial x_1^p}{\partial p}\frac{\partial AC^*}{\partial w_1}.$$ Multiplying both sides of ♦ by w_1/x_1^* yields:

♣ $$\frac{w_1}{x_1^*}\frac{\partial x_1^*}{\partial w_1} \equiv \frac{w_1}{x_1^*}\frac{\partial x_1^p}{\partial w_1} + \frac{w_1}{x_1^*}\frac{\partial x_1^p}{\partial p}\frac{\partial AC^*}{\partial w_1}.$$ Now applying the envelope theorem result from c,

$\frac{\partial AC^*}{\partial w_1} = \frac{x_1^*}{y^*}$, and multiplying by *p/p* (one) yields: $\frac{w_1}{x_1^*}\frac{\partial x_1^p}{\partial p}\frac{\partial AC^*}{\partial w_1} = \frac{p}{x_1^*}\frac{\partial x_1^p}{\partial p}\frac{w_1 x_1^*}{py^*}$. Thus, it follows that:

$$\frac{w_1}{x_1^*}\frac{\partial x_1^*}{\partial w_1} \equiv \frac{w_1}{x_1^*}\frac{\partial x_1^p}{\partial w_1} + \frac{p}{x_1^*}\frac{\partial x_1^p}{\partial p}\frac{w_1 x_1^*}{p^* y^*}$$ or $$\frac{\partial \ln x_1^*}{\partial \ln w_1} \equiv \frac{\partial \ln x_1^p}{\partial \ln w_1} + \frac{\partial \ln x_1^p}{\partial \ln p}\frac{w_1 x_1^*}{p^* y^*},$$ which is the desired result.

Comments:

1. For Question 3 part B).

① If you would like to show it in a general case (disregard cobb-douglas production function)

To setup the SRC: $\min C = w_1 x_1 + w_2 \tilde{x}_2$ s.t. $y = f(x_1, \tilde{x}_2)$

$\Rightarrow C(w_1, w_2, x_1, \tilde{x}_2)$ or $C(w_1, w_2, y, \tilde{x}_2)$ or $C(w_1, w_2, y, \tilde{x}_2(y))$.

To note that you shouldn't setup C as $C(w_1, w_2, y, x_1, \tilde{x}_2)$ or $C(w_1, w_2, y, x_1(y), \tilde{x}_2(y))$

In the same case, to setup the SRAC, $SRAC = \frac{SRC}{y}$.

$\Rightarrow SRAC(w_1, w_2, y, \tilde{x}_2(y))$

Method I — use C, not AC

Let y^* be some given level of output, and let $\tilde{x}_2^* = \tilde{x}_2(y^*)$ be the associated long-run demand for the fixed factor $\tilde{x}_2$. The short-run cost, $c(w_1, w_2, y, \tilde{x}_2^*)$ must be at least as great as the long-run cost, $c(y, \tilde{x}_2(y))$ for all levels of output, and the short-run cost will equal the long-run cost at output y^*, so $\overset{\text{short}}{c}(y^*, \tilde{x}_2^*) = \overset{\text{long}}{c}(y^*, \tilde{x}_2(y^*))$.

Mathematically, $$\frac{dc(y^*, \tilde{x}_2(y^*))}{dy} = \frac{\partial c(y^*, \tilde{x}_2^*)}{\partial y} + \frac{\partial c(y^*, \tilde{x}_2^*)}{\partial \tilde{x}_2} \frac{\partial \tilde{x}_2(y^*)}{\partial y}.$$

Since $\tilde{x}_2^*$ is the optimal choice of the fixed factors at the output level y^*, we must have

$$\frac{\partial c(y^*, \tilde{x}_2^*)}{\partial \tilde{x}_2} = 0$$

$$\Rightarrow \frac{dc(y^*, \tilde{x}_2(y^*))}{dy} = \frac{\partial c(y^*, \tilde{x}_2^*)}{\partial y},$$ or LRMC at y^* equal SRMC at $(y^*, \tilde{x}_2^*)$

Next, if LR and SR cost curves are tangent, the LR and SR average cost curves must also be tangent at the same point. And since LRAC is the lower envelope of the SRAC, the conclusion follows min SAC = LAC.

Method II — Use AC directly.

$$\frac{d\,AC(y^*, \bar{x}_2(y^*))}{dy} = \frac{\partial AC(y^*, \bar{x}_2^*)}{dy} + \frac{\partial AC(y^*, \bar{x}_2^*)}{\partial \bar{x}_2}\frac{\partial \bar{x}_2(y^*)}{\partial y}$$

Again, since $\bar{x}_2^*$ is fixed in the short run.

$$\Rightarrow \frac{\partial AC(y^*, \bar{x}_2^*)}{\partial \bar{x}_2} = 0$$

$$\Rightarrow \frac{d\,AC(y^*, \bar{x}_2(y^*))}{dy} = \frac{\partial AC(y^*, \bar{x}_2^*)}{dy}.$$

$\Rightarrow$ SRAC and LRAC are tangent.

② However, both methods do not use the specific production function. You are supposed to prove the results in this specific case

2. Like what I said in the discussion section, you can prove Question 1 part (B) In the general approach:

Let x_1, x_2, x_3 are the factor demand under the price of w_1, w_2, w_1+w_2, using cost minimizing approach.

Therefore we have

$$c(w_1, y) = w_1 x_1.$$

$$c(w_2, y) = w_2 x_2$$

$$c(w_1+w_2, y) = (w_1+w_2)\,x_3.$$

By the definition of minimum cost of $c(w_1, y)$, $c(w_2, y)$, we have

$$w_1 x_1 \le w_1 x_3, \quad w_2 x_2 \le w_2 x_3.$$

$$\Rightarrow \quad c(w_1+w_2, y) = (w_1+w_2)x_3 = w_1x_3 + w_2x_3 \geqslant w_1x_1 + w_2x_2$$

$$= c(w_1, y) + c(w_2, y).$$

(QED.)