Question 1.

(1) $\max_{x_1, x_2} \; p(\alpha_1 \ln x_1 + \alpha_2 \ln x_2) - w_1 x_1 - w_2 x_2$.

FONC: $\begin{cases} \dfrac{p\alpha_1}{x_1} - w_1 = 0 \\ \dfrac{p\alpha_2}{x_2} - w_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = \dfrac{p\alpha_1}{w_1} \\ x_2 = \dfrac{p\alpha_2}{w_2} \end{cases}$

Need to check for sosc.

Hessian $\left.\begin{pmatrix} -p\alpha_1 / x_1^2 & 0 \\ 0 & -p\alpha_2 / x_2^2 \end{pmatrix}\right|_{x_1 = \frac{p\alpha_1}{w_1},\ x_2 = \frac{p\alpha_2}{w_2}}$

sosc: $\begin{cases} -p\alpha_1 \left(\dfrac{w_1}{p\alpha_1}\right)^2 = -\dfrac{w_1^2}{p\alpha_1} < 0 \\ p^2 \alpha_1 \alpha_2 \left(\dfrac{w_1}{p\alpha_1}\right)^2 \left(\dfrac{w_2}{p\alpha_2}\right)^2 = \dfrac{w_1^2 w_2^2}{\alpha_1 \alpha_2} > 0 \end{cases}$

$\Rightarrow$ Negative Definiteness holds.

Actually it's also concave, since

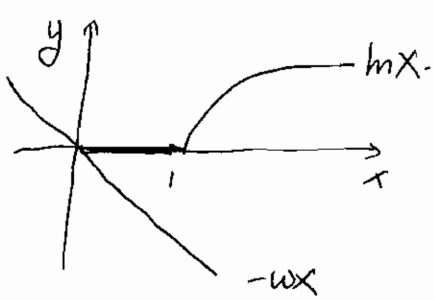
$\begin{cases} \dfrac{-p\alpha_1}{x_1^2} < 0 \\ \dfrac{-p\alpha_2}{x_2^2} < 0 \\ \dfrac{-p\alpha_1}{x_1^2} \cdot \dfrac{-p\alpha_2}{x_2^2} - 0 = \dfrac{p^2 \alpha_1 \alpha_2}{x_1^2 x_2^2} > 0. \end{cases}$

$\Rightarrow$ Negative Semi-Definiteness at $\forall (x_1, x_2)$ in the domain holds.

$\Rightarrow x_1^* = \dfrac{p\alpha_1}{w_1}, \quad x_2^* = \dfrac{p\alpha_2}{w_2} \Rightarrow y = \alpha_1 \ln \dfrac{\alpha_1 p}{w_1} + \alpha_2 \ln \dfrac{\alpha_2 p}{w_2}$

$\pi = p\left(\alpha_1 \ln \dfrac{\alpha_1 p}{w_1} + \alpha_2 \ln \dfrac{\alpha_2 p}{w_2}\right) - \alpha_1 - \alpha_2$.

(2).



$\pi(x) = pf(x) - wx = p\ln x - wx \quad \text{if } x > 1 \quad ①$

$\pi'(x) = \dfrac{p}{x} - w = 0 \Rightarrow x = \dfrac{p}{w} \quad \text{if } x > 1 \quad ②$.

$\Rightarrow$ $\pi(x)$ is increasing at $(-\infty, \frac{P}{\omega}]$, and decreasing at $[\frac{P}{\omega}, +\infty)$

Then we need to consider the following cases.

① If $\frac{P}{\omega} \leq 1$, then the maximum profit the firm could earn is at $x=1$, where $\pi(1) = -\omega < 0$. So the firm should choose not to produce anything.

② If $P/\omega > 1$, the maximum profit the firm could earn is at $x = P/\omega$, where $\pi(P/\omega) = P[\ln(P/\omega) - 1]$. Therefore the firm should choose not to produce anything if $P/\omega \leq e$; otherwise the firm should choose to produce exactly $x = P/\omega$ to earn positive profits.

$$\Rightarrow \pi(p, \omega) = \begin{cases} 0, & \text{if } P/\omega \leq e \\ P[\ln(P/\omega) - 1], & \text{if } P/\omega > e. \end{cases}$$

Comments:

① profit/ production functions should be a function of (p, ω), since firms are making decisions based on exogenous (p, ω).

Some of you have answers like: $\pi = P[\ln(P/\omega) - 1]$ if $x > 1$.

You need to furtherly figure out when the firms will produce $x > 1$, which means $P/\omega > 1$ should be achieved as the first step; then based on the profit function, $P/\omega \geq e$ is also needed to ensure positive profits.

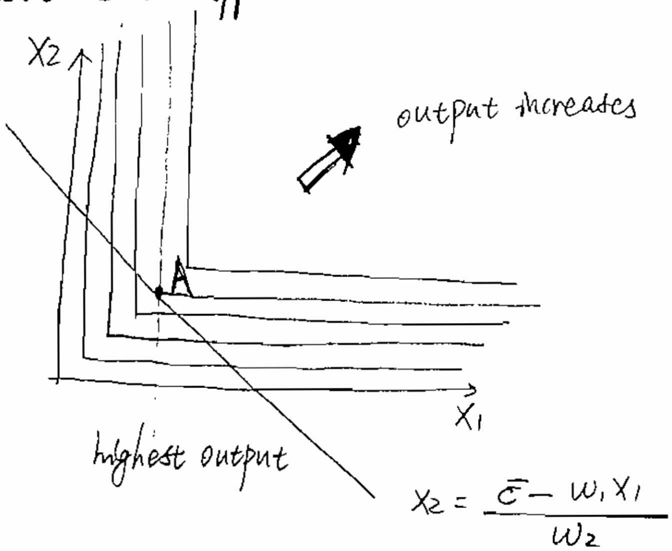
② One thing to ~~keep~~ be mindful is that firms shouldn't earn negative profits. They can simply earn zero and get better compared with negative profits.

(3). $\pi = \max \; p \min\{x_1, x_2\}^a - w_1 x_1 - w_2 x_2$

The firm will end up using the same amount of x_1 and x_2.

In order to get this conclusion, you can follow two ways.

① Level set Approach



You can draw the isoquant curve (level set of the production function). As the output increases, the ~~tex~~ isoquant curve moves to the upper-right direction. The cost structure is fixed: $X_2 = -\frac{w_1}{w_2} X_1 + \frac{\bar{c}}{w_2}$. Set $\bar{c}$ fixed, so the cost curve doesn't move. You can see that the highest output (profits) is achieved when the isoquant curve touches the cost curve at point A, where $X_1 = X_2$.

② FOC Approach. (Some of you are using this way. It's smart).

Let $X_1 > X_2$, then $\pi = \max \; p \cdot X_2^a - w_1 X_1 - w_2 X_2$.

FOC: $\frac{\partial \pi}{\partial X_2} = p a X_2^{a-1} - w_2 = 0$

$\frac{\partial \pi}{\partial X_1} = -w_1 = 0 \Rightarrow$ which cannot be true.

Since FOC doesn't hold, this cannot be a local maximum point.

Let $X_2 > X_1$, then $\pi = \max \; p X_1^a - w_1 X_1 - w_2 X_2$

FOC: $\frac{\partial \pi}{\partial X_2} = -w_2 = 0 \Rightarrow$ which cannot be true.

Since FOC doesn't hold, this cannot be a local maximum point

We are left with $X_1 = X_2$.

$x_1 = x_2 \Rightarrow \pi = \max\, p x^a - (w_1 + w_2) x$

FONC: $\quad p a x^{a-1} = w_1 + w_2$

$$\Rightarrow x_1 = x_2 = \left(\frac{w_1 + w_2}{pa}\right)^{\frac{1}{a-1}}$$

check for SOSC.

$$\pi_{xx} = pa(a-1)x^{a-2}\Big|_{x=\left(\frac{w_1+w_2}{pa}\right)^{\frac{1}{a-1}}} = pa(a-1)\cdot\left(\frac{w_1+w_2}{pa}\right)^{\frac{a-2}{a-1}}$$

If $0 < \alpha < 1$, $\pi_{xx} < 0$, we'll have maximum point.

If $\alpha \geq 1$, $\pi_{xx} \geq 0$, SOSC doesn't hold $\Rightarrow$ Increasing / constant return to scale

plugging on $x_1^* = x_2^* = \left(\frac{w_1+w_2}{pa}\right)^{\frac{1}{a-1}}$, you'll get the ~~prof~~ profit and production functions.

(4) $\quad f(x_1, x_2) = \frac{1}{2}x_1^2 + \ln x_2$.

$$\pi = p\left(\frac{1}{2}x_1^2 + \ln x_2\right) - w_1 x_1 - w_2 x_2$$

FONC $\begin{cases} x_1 p - w_1 = 0 \\ \frac{p}{x_2} - w_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{w_1}{p} \\ x_2 = \frac{p}{w_2} \end{cases}$

check for SOSC: Hessian $\begin{pmatrix} p & 0 \\ 0 & -\frac{p}{x_2^2} \end{pmatrix}\Bigg|_{\substack{x_1 = w_1/p \\ x_2 = p/w_2}}$.

since $\begin{cases} p > 0, \\ p\cdot\left(-\frac{p}{x_2^2}\right) - 0 = -\frac{p^2}{\frac{p^2}{w_2^2}} = -w_2^2 < 0. \end{cases}$

It's indefinite at $(w_1/p, \ p/w_2)$.

We cannot find an optimal point.

why?

Marginal product per dollar input for x_1: $\frac{x_1}{w_1}$, increasing with x_1

Marginal product per dollar input for x_2: $\frac{1}{w_2 x_2}$, decreasing with x_2

$\Rightarrow$ When x_2 is positively close to zero, its marginal product per dollar is greater than x_1. Firms have incentives to use more of x_2, instead of x_1. However, up to certain points, the marginal product per dollar of x_1 will be greater than x_2, and the firm will use more of x_1 instead of x_2. Since we do not put any cost constraint here, the firm could use as many x_1 as possible to achieve more and more profits.

Comments:

① For part (2), be mindful that the production function is not concave. Thus you need to specify the two cases and have corner solutions there.

②. For part (3), be mindful of the meaning of FONC.

Question 2:

A. $Y = f(X_1, X_2) = 24X_1 + 14X_2 - X_1^2 + X_1X_2 - X_2^2$

$f(X_1, X_2)$ is a polynomial $\Rightarrow$ continuous

2pts.
1pt. Marginals
1pt. IFT

Using the Implicit Function Theorem

$$\frac{\partial X_2}{\partial X_1} = -\frac{f_1}{f_2} = -\frac{(24 - 2X_1 + X_2)}{14 + X_1 - 2X_2} \quad \text{s.t. } f_2 \neq 0$$

$$= \frac{2X_1 - 24 - X_2}{14 + X_1 - 2X_2} \quad \text{s.t. } f_2 \neq 0$$

B. $\pi(p, w_1, w_2, X_1, X_2) = pf(X_1, X_2) - w_1X_1 - w_2X_2$

$$\pi = p(24X_1 + 14X_2 - 2X_1^2 + X_1X_2 - 2X_2^2) - w_1X_1 - w_2X_2$$

$\max\limits_{X_1, X_2} \pi(p, \mathbf{w}, \mathbf{X})$ s.t. $X_1 \geq 0, X_2 \geq 0$

5 pts
1pt Set up
2pt FOC's & solving for X^*
1pt SOSC
1pt Y^*

FIRST ORDER CONDITIONS

$$\frac{\partial \pi}{\partial X_1} = p(24 - 2X_1 + X_2) - w_1 \equiv 0 \quad (a)$$

$$\frac{\partial \pi}{\partial X_2} = p(14 + X_1 - 2X_2) - w_2 \equiv 0 \quad (b)$$

Rearranging (a) and solving for X_1, we get.

$$X_1 = \frac{24p - w_1 + pX_2}{2p} = 12 + \frac{X_2}{2} - \frac{w_1}{2p} \quad (c)$$

Substituting (c) into (b)

$$14 + 12 + \frac{X_2}{2} - \frac{w_1}{2p} - 2X_2 = \frac{w_2}{p}$$

Solve for $x_2^* = \frac{52}{3} - \left(\frac{2w_2 + w_1}{3p}\right)$ (d)

Substituting (d) back into (c), we get:

$$x_1^* = 12 - \frac{w_1}{2p} + \frac{1}{2}\left(\frac{52}{3} - \left(\frac{2w_2 + w_1}{3p}\right)\right)$$

After simplifying:

$$x_1^* = \frac{62}{3} - \left(\frac{2w_1 + w_2}{3p}\right) \quad \text{(e)}$$

<u>Checking Second Order Conditions</u>

$f_{11} = -2$

$f_{22} = -2$

$f_{12} = f_{21} = 1$

$\pi_{11} = pf_{11}$

$\pi_{22} = pf_{22}$

$\pi_{12} = \pi_{21} = pf_{12}$

by Young's Theorem

$$H = p\begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix} = p\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\left.\begin{array}{l} |H| = 4p^2 - p^2 = 3p^2 > 0 \\ pf_{11} = -2p < 0 \\ pf_{22} = -2p < 0 \end{array}\right\} \Rightarrow$$

The Hessian Matrix is negative definite. for all x.
The SOSC's are satisfied.

Thus, the factor demand curves derived from the FOC's are an interior solution.

To get the optimal y^* substitute (d) & (e) into $f(x_1, x_2)$

$$y^*(p, w_1, w_2) = 24\left(\frac{62p - 2w_1 - w_2}{3p}\right) + 14\left(\frac{52p - w_1 - 2w_2}{3p}\right)$$
$$- \left(\frac{62p - 2w_1 - w_2}{3p}\right)^2 + \left(\frac{62p - 2w_1 - w_2}{3p}\right)\left(\frac{52p - w_1 - 2w_2}{3p}\right)$$
$$- \left(\frac{52p - 2w_2 - w_1}{3p}\right)^2$$

Simplifying the expression

$$y^*(p, w_1, w_2) = \frac{1108}{3} - \left(\frac{w_1^2 + w_1 w_2 + w_2^2}{3p^2}\right)$$

C. Recall $H = p\begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix} = p\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

<u>4pts</u>

1.5 pts. set up
1.5 pt. Comp Stat
1 pt. Description

FOC's:

$$p f_1(x_1, x_2, w, p) \equiv w_1$$

$$p f_2(x_1, x_2, w, p) \equiv w_2$$

with our given function: (From part B)

$$p(24 - 2x_1 + x_2) \equiv w_1 \qquad (g)$$

$$p(14 + x_1 - 2x_2) \equiv w_2 \qquad (h)$$

Differentiate (g) & (h) w/ respect to w_1 and then w/ respect to w_2

$$-2p\frac{\partial x_1^*}{\partial w_1} + p\frac{\partial x_2^*}{\partial w_1} = 1 \qquad -2p\frac{\partial x_1^*}{\partial w_2} + p\frac{\partial x_2^*}{\partial w_2} = 0$$

$$p\frac{\partial x_1^*}{\partial w_1} + -2p\frac{\partial x_2^*}{\partial w_1} = 0 \qquad p\frac{\partial x_1^*}{\partial w_2} + -2p\frac{\partial x_2^*}{\partial w_2} = 1$$

In Matrix Notation:

$$p\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \frac{\partial x_1^*}{\partial w_1} & \frac{\partial x_1^*}{\partial w_2} \\ \frac{\partial x_2^*}{\partial w_1} & \frac{\partial x_2^*}{\partial w_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}}_{= H}$$

$|H| = 3$ (From part B)

$$H^{-1} = \frac{\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}}{3} = \begin{bmatrix} -2/3 & -1/3 \\ -1/3 & -2/3 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial x_1^*}{\partial w_1} & \frac{\partial x_1^*}{\partial w_2} \\ \frac{\partial x_2^*}{\partial w_1} & \frac{\partial x_2^*}{\partial w_2} \end{bmatrix} = \frac{1}{p} H^{-1} = \frac{1}{p}\begin{bmatrix} -2/3 & -1/3 \\ -1/3 & -2/3 \end{bmatrix}$$

$$\left.\begin{aligned} \frac{\partial x_1^*}{\partial w_1} &= \frac{-2}{3p} < 0 \\ \frac{\partial x_2^*}{\partial w_2} &= \frac{-2}{3p} < 0 \end{aligned}\right\}$$

These are both less than zero. As the price of an input factor ↑, the quantity of the input factor ↓.

$$\frac{\partial x_1^*}{\partial w_2} = \frac{\partial x_2^*}{\partial w_1} = \frac{-1}{3p} < 0$$ & SYMMETRIC

In <u>this</u> case, if the input price of one factor ↑, the quantity (used in the production process) of <u>Both</u> x_1 and x_2 will decrease.

D. $\frac{\partial x_1^*}{\partial p} = \frac{2w_1 + w_2}{3p^2} > 0$

$\frac{\partial x_2^*}{\partial p} = \frac{w_1 + 2w_2}{3p^2} > 0$

As the output price $\uparrow$, the use of both factor inputs $\uparrow$.

$\underline{3 \text{pts}}$

2 pts. $\frac{\partial x}{\partial p}$

1 pt weighted AV.

NOTE from FOC's:

$$\left.\begin{aligned} p f_1(x_1, x_2, W, p) &= w_1 \\ p f_2(x_1, x_2, W, p) &= w_2 \end{aligned}\right\} \Rightarrow \begin{aligned} f_1 &= \frac{w_1}{p} \\ f_2 &= \frac{w_2}{p} \end{aligned}$$

$$f_1 \frac{\partial x_1}{\partial p} + f_2 \frac{\partial x_2}{\partial p} = \frac{w_1}{p}\left(\frac{2w_1 + w_2}{3p^2}\right) + \frac{w_2}{p}\left(\frac{w_1 + 2w_2}{3p^2}\right)$$

$$= \frac{2(w_1 + w_2)}{3p^3} > 0 \text{ if } w_1, w_2, p > 0.$$

E. $x_1^*(\lambda p, \lambda w_1, \lambda w_2) = \frac{62}{3} - \left(\frac{2\lambda w_1 + \lambda w_2}{3\lambda p}\right)$

$= \frac{62}{3} - \left(\frac{2w_1 + w_2}{3p}\right)$

$= x_1^*(p, w_1, w_2) \Rightarrow$ HOD zero

$\underline{3 \text{pts}}$

½ pt. setup

2 pt HOD x_1 & x_2

½ pt Description/Interpretation

$x_2^*(\lambda p, \lambda w_1, \lambda w_2) = \frac{52}{3} - \left(\frac{\lambda w_1 + 2\lambda w_2}{3\lambda p}\right)$

$= \frac{52}{3} - \left(\frac{w_1 + 2w_2}{3p}\right)$

$= x_2^*(p, w_1, w_2) \Rightarrow$ HOD zero

Since x_1^* & x_2^* are HOD zero in p, w_1, and w_2, it means that the factor demand functions are not affected if ALL prices are Δed by the same amount. Only changes in the relative prices ~~matters~~.

(F) From our identities 2pts ½pt setup ½ interpretation 1 pt $\frac{\partial y^*}{\partial p}$

$\frac{\partial y^*}{\partial p} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial p}$ which is the weighted average of $\frac{\partial x_1}{\partial p}$ & $\frac{\partial x_2}{\partial p}$ calculated in part D.

$$\Rightarrow \frac{\partial y^*}{\partial p} = \frac{2(w_1+w_2)^2}{3p^3} > 0$$

As $p \uparrow$, the optimal amount of output produced should $\uparrow$.
the outputprice

G. From the identities

3pts

2pts $\frac{\partial y^*}{\partial w}$

1pt ~~own~~ elasticity

$$\frac{\partial y^*}{\partial w_1} = \frac{-\partial x_1^*}{\partial p} = -\left(\frac{2w_1+w_2}{3p^2}\right)$$

$$\frac{\partial y^*}{\partial w_2} = \frac{-\partial x_2^*}{\partial p} = -\left(\frac{w_1+2w_2}{3p^2}\right)$$

$$\frac{\partial y^*}{\partial w_1}\cdot\frac{w_1}{y^*} + \frac{\partial y^*}{\partial w_2}\cdot\frac{w_2}{y^*} + \frac{\partial y^*}{\partial p}\cdot\frac{p}{y^*} = \frac{1}{y^*}\cdot\left[\frac{-2w_1^2+-w_2w_1-w_1w_2-2w_2^2+2(w_1+w_2)^2}{3p^2}\right]$$

$$= \frac{0}{3p^2y^*}$$