

**Homework #4**  
(Due 11/10/2009)

1. Given production function  $y = x_1^{\alpha_1} x_2^{\alpha_2}$  where  $\alpha_1 + \alpha_2 = 1$ , and input prices  $w_i, i = 1, 2$ .
  - a) Show that the constant-output (*i.e. cost minimizing*) factor demand function have the form  $x_i^*(w_1, w_2, y) = k_i w_i^{\frac{-\alpha_j}{\alpha_1 + \alpha_2}} w_j^{\frac{\alpha_j}{\alpha_1 + \alpha_2}} y^{\frac{1}{\alpha_1 + \alpha_2}}$ , where  $i \neq j$ , and  $k_i$  is a constant.
  - b) Show that the cost function has the form
 
$$C^*(w_1, w_2, y) = (k_1 + k_2) w_1^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} w_2^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} y^{\frac{1}{\alpha_1 + \alpha_2}} .$$
  - c) Verify that Shephard's Lemma holds,  $\frac{\partial C^*}{\partial w_i} = x_i^*$ .
  - d) Show that the share of spending on each cost minimizing input is independent of output level and input prices.
  
2. (Varian p.79, question 5.16) For each cost function below, determine if it is homogeneous of degree one, monotonic and concave in  $\mathbf{w}$ , and/or continuous. If it is, derive the associated production function (hint: you may refer to the example on page 87 for some help).
  - (a)  $c(\mathbf{w}, y) = y^{\frac{1}{2}} (w_1 w_2)^{\frac{3}{4}}$
  - (b)  $c(\mathbf{w}, y) = y(w_1 + \sqrt{w_1 w_2} + w_2)$
  - (c)  $c(\mathbf{w}, y) = y(w_1 e^{-w_1} + w_2)$
  - (d)  $c(\mathbf{w}, y) = y(w_1 - \sqrt{w_1 w_2} + w_2)$
  - (e)  $c(\mathbf{w}, y) = (y + \frac{1}{y}) \sqrt{w_1 w_2}$
  
- 3.. Consider the average cost  $AC = \frac{(w_1 x_1 + w_2 x_2)}{y}$  where  $x_1$  and  $x_2$  are factor inputs,  $w_1$  and  $w_2$  are factor prices, and  $y = g(x_1, x_2)$  is the production function. Consider the minimization problem
 
$$AC^*(w_1, w_2) = \underset{\mathbf{x}, y}{\text{Min}} \left\{ \frac{w_1 x_1 + w_2 x_2}{y} : y = g(\mathbf{x}) \right\} ,$$
 which has for solution  $x_i^*(w_1, w_2)$ ,  $i = 1, 2$ , and  $y^*(w_1, w_2)$ , where  $AC^*(w_1, w_2)$  is the minimum average cost for given factor prices.
  - a) Explain how the factor demands  $x_i^*(w_1, w_2)$  and the indirect objective function  $AC^*(w_1, w_2)$  are derived. Prove that the factor demands are homogeneous of degree zero and that  $AC^*$  is homogeneous of degree one in factor prices.
  - b) What is the slope of  $AC^*$  at a given  $w_1$ ?
  - c) Show that  $\frac{\partial (\frac{x_i^*}{y^*})}{\partial w_i} < 0$ . (hint: use the dual approach to show  $AC^*$  is concave in  $\mathbf{w}$ )
  - d) Show that the elasticity of demand for factor 1 is less than the elasticity of output supply with respect to  $w_1$ .
  - e) Set up the primal-dual model, minimize  $AC - AC^*$ , and derive the above results.
  - f) Contrast the factor demands derived from this model  $x_i^*(w_1, w_2)$  with the profit-maximizing input demands  $x_i^p(w_1, w_2, p)$  for a competitive firm. Display the first-

order conditions for both models (note that you are not asked to solve the problem), and explain the relation between the models by using the following identities:

$$x_i^*(w_1, w_2) \equiv x_i^P(w_1, w_2, p^*(w_1, w_2)), i = 1, 2,$$

where  $p^*(w_1, w_2) \equiv AC^*(w_1, w_2)$ .

- (h) From these identities, show that the elasticity of demand for  $x_1$  derived from minimizing average cost,  $\frac{\partial x_1^*}{\partial w_1} \cdot \frac{w_1}{x_1^*}$ , is equal to the elasticity of demand derived from profit maximization, plus an output effect which equals the share spent on  $x_1$  times the output price elasticity of  $x_1$ .