

Question 1: Properties of Cost Functions and Conditional Factor Demand Functions

A. Consider a firm with conditional factor demand functions of the form

$$x_1 = 1 + 3w_1^{-1/2}w_2^a$$

$$x_2 = 1 + bw_1^{1/2}w_2^c$$

Output has been set equal to 1 for convenience. What are the values of the parameters a, b, and c and why? Explain and, in particular, state all the conditions you use.

B. A real-valued function is called “**superadditive**” if $f(\mathbf{z}_1 + \mathbf{z}_2) \geq f(\mathbf{z}_1) + f(\mathbf{z}_2)$. Show that every cost function is “superadditive” in input prices. Use this to prove that the cost function is non-decreasing in input prices WITHOUT requiring it to be differentiable.

C. True or False: The cost function $c(w_1, w_2, y) = (\sqrt{w_1} + \sqrt{w_2})y$ is an impostor (some other function pretending to be a cost function). Justify your answer.

D. Calculate the cost function and the conditional input demands for the linear production function $y = \sum_i \alpha_i x_i$.

Question 2: CES Technology

There are two factors of production and the production function is $f(x_1, x_2) = (x_1^{1/2} + x_2^{1/2})^2$.

A. Calculate $x_1(w_1, w_2, y)$ and $c(w_1, w_2, y)$.

B. Derive the elasticity of substitution and the Allen Elasticity of Substitution for this production function.

C. Verify that Shepard’s Lemma holds

D. Suppose that a competitive firm faces the wages (w_1, w_2) for its inputs. How does the ratio of factor shares

$$\frac{w_1 x_1(w_1, w_2, y)}{w_2 x_2(w_1, w_2, y)} \text{ vary with the ratio of } \left(\frac{w_1}{w_2} \right)?$$

Question 3: Long Run versus Short Run Cost Functions

Derive the cost function for the two-input, constant-returns, Cobb-Douglas technology. Fix one input (x_2) and derive the short-run cost function.

A. Show that long-run average and long-run marginal cost are constant and equal.

B. Show that for every level of the fixed input, short-run average cost and long-run average cost are equal at the minimum level of short-run average cost.

C. Illustrate your results in the cost-output plane.

Question 4: Cost Minimization and Duality

Consider the average cost $AC = \frac{(w_1x_1 + w_2x_2)}{y}$ where x_1 and x_2 are factor inputs, w_1 and w_2 are factor prices, and $y = g(x_1, x_2)$ is the production function. Consider the minimization problem

$$AC^*(w_1, w_2) = \underset{\mathbf{x}, y}{\text{Min}} \left\{ \frac{w_1x_1 + w_2x_2}{y} : y = g(\mathbf{x}) \right\},$$

which has for solution $x_i^*(w_1, w_2)$, $i = 1, 2$, and $y^*(w_1, w_2)$, where $AC^*(w_1, w_2)$ is the minimum average cost for given factor prices.

- A. Explain how the factor demands $x_i^*(w_1, w_2)$ and the indirect objective function $AC^*(w_1, w_2)$ are derived. Prove that the factor demands are homogeneous of degree zero and that AC^* is homogeneous of degree one in factor prices.
- B. On a graph with AC and AC^* on the vertical axis, and w_1 on the horizontal axis, plot a typical AC and AC^* . Show graphically that AC^* is necessarily concave in w_1 (and of course in w_2 also).
- C. What is the slope of AC^* at a given w_1 ?
- D. Using this graphical analysis, show that $\frac{\partial(\frac{x_i^*}{y^*})}{\partial w_i} < 0$.
- E. Show that the elasticity of demand for factor 1 is less than the elasticity of output supply with respect to w_1 .
- F. Set up the primal-dual model, minimize $AC - AC^*$, and derive the above results algebraically.
- G. Contrast the factor demands derived from this model $x_i^*(w_1, w_2)$ with the profit-maximizing input demands $x_i^p(w_1, w_2, p)$ for a competitive firm. Display the first-order conditions for both models, and explain the relation between the models by using the following identities:

$$x_i^*(w_1, w_2) \equiv x_i^p(w_1, w_2, p^*(w_1, w_2)), i = 1, 2,$$

$$\text{where } p^*(w_1, w_2) \equiv AC^*(w_1, w_2).$$

- H. From these identities, show that the elasticity of demand for x_1 derived from minimizing average cost, $\frac{\partial x_1^*}{\partial w_1} \cdot \frac{w_1}{x_1^*}$, is equal to the elasticity of demand derived from profit maximization, plus an output effect which equals the share spent on x_1 times the output price elasticity of x_1 .