Question 1: Economic Application and Verification of LeChatelier's Principle (Silberberg: Page 89 Question 9)

Assumptions:

1. Firm is profit maximizing.
2. The firm's profit function is strictly concave (SOSC for profit maximization).
3. Input and output prices are greater than 0.

Part A: Derive the factor demand functions. Are these choice function homogeneous of any degree in any of the parameters?

2 points Total 1pt. Set up and Factor Demand 1 pt. Homogeneity Note: You needed to prove that adding a tax to the profit maximization did not change homogeneity of the factor demand

$$\max_{x_1,x_2} \pi\left(x_1, x_2, w_1, w_2, p, t\right) = pf(x_1, x_2) - w_1x_1 - w_2\left(1+t\right)x_2$$

$$\pi_1 = pf_1(x_1, x_2) - w_1$$

$$\pi_2 = pf_2(x_1, x_2) - w_2 - tw_2$$

$FONC$:

$$pf_1(x_1, x_2) - w_1 = 0$$

$$pf_2(x_1, x_2) - w_2\left(1+t\right) = 0$$

We get the input factor demand curves, $x_1 = x_1^*(w_1, w_2, p, t)$ and $x_2 = x_2^*(w_1, w_2, p, t)$ if the SOSC holds.

Note: Since $\pi_{ij} = pf_{ij}$, the second order sufficiency condition for profit maximization reduce down to

$$\begin{aligned} &pf_{11}, pf_{22} < 0 \\ &p^2 f_{11} f_{22} - p^2 f_{12}^2 > 0 \end{aligned}.$$

The input factor demand functions will be homogenous of degree zero in p, w_1, w_2. To see this note, that a firm facing prices kp, kw_1, and kw_2, such that k > 0 solves: $\max_{x_1,x_2} \pi\left(x_1, x_2, kw_1, kw_2, kp, t\right) = kpf(x_1, x_2) - kw_1x_1 - kw_2\left(1+t\right)x_2$ which is equivalent to solving the following: $\max_{x_1,x_2} k\pi\left(x_1, x_2, w_1, w_2, p, t\right) = k\left\{pf(x_1, x_2) - w_1x_1 - w_2\left(1+t\right)x_2\right\}$. Since the profit function is HOD 1, it follows that the first order conditions will be HOD zero. That is to say, $x_1^*(w_1, w_2, p, t) = x_1^*(kw_1, kw_2, kp, t)$ and $x_2^*(w_1, w_2, p, t) = x_2^*(kw_1, kw_2, kp, t)$.

Part B: Show that if the tax rate rises, the firm will use less of factor 2.

2 Points Total

Differentiate π_1 and π_2 (the first order conditions) with respect to t:

$$pf_{11}\frac{dx_1}{dt} + pf_{12}\frac{dx_2}{dt} = 0$$

$$pf_{21}\frac{dx_1}{dt} + pf_{22}\frac{dx_1}{dt} = w_2$$

In Matrix Notation $\begin{bmatrix} pf_{11} & pf_{12} \\ pf_{12} & pf_{22} \end{bmatrix}\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ w_2 \end{bmatrix}$

Solve for $\dfrac{dx_1}{dt}$ and $\dfrac{dx_2}{dt}$ using Cramer's Rule or the inverse. $\dfrac{dx_1}{dt} = \dfrac{-f_{12}w_2}{p(f_{11}f_{22} - f_{12}^2)}$ and $\dfrac{dx_2}{dt} = \dfrac{f_{11}w_2}{p(f_{11}f_{22} - f_{12}^2)}$.

Due to the second order sufficiency conditions of profit maximization, $(f_{11}f_{22} - f_{12}^2) > 0$, f_{22}, $f_{11} < 0$, $\dfrac{dx_2}{dt} < 0$. Therefore, as the tax increases, the firm will use less of factor 2.

Part C: Show that $\frac{\partial x_1 *}{\partial t} = w_2 \frac{\partial x_2 *}{\partial w_1}$

First we need to find the response of x to a change in w.

2 Points Total

Differentiate π_1 *and* π_2 *with respect to* w_1*:*

$$pf_{11}\frac{dx_1}{dw_1} + pf_{12}\frac{dx_2}{dw_1} - 1 = 0$$

$$pf_{21}\frac{dx_1}{dw_1} + pf_{22}\frac{dx_2}{dw_1} = 0$$

$$pf_{11}\frac{dx_1}{dw_1} + pf_{12}\frac{dx_2}{dw_1} = 1$$

$$pf_{21}\frac{dx_1}{dw_1} + pf_{22}\frac{dx_2}{dw_1} = 0$$

In Matrix Notation: $\begin{bmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{bmatrix} \begin{bmatrix} \frac{dx_1}{dw_1} \\ \frac{dx_2}{dw_1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solve for $\frac{dx_1}{dw_1}$ and $\frac{dx_2}{dw_1}$. $\frac{dx_1}{dw_1} = \frac{f_{22}}{p(f_{11}f_{22} - f_{12}^2)}$ $\frac{dx_2}{dw_1} = \frac{-f_{21}}{p(f_{12}^2 - f_{11}f_{22})}$

Therefore, $(w_2)\frac{dx_2}{dw_1} = \frac{-w_2 f_{21}}{p(f_{12}^2 - f_{11}f_{22})}$. Recall from **part B** that $\frac{dx_1}{dt} = \frac{-f_{12}w_2}{p(f_{11}f_{22} - f_{12}^2)}$. Since $f_{12} = f_{21}$, $\frac{dx_1}{dt} = (w_2)\frac{dx_2}{dw_1}$.

Part D: Suppose that factor 1 is held fixed at its profit-maximizing level. Show that the response of factor 2 to a change in the tax rate is less (in absolute value) than before.

Suppose factor 1 is held fixed in the short run. $x_1 = \overline{x}_1$

$$\max_{x_2} \pi = pf(\overline{x}_1, x_2(p, w_2, w_1, t)) - w_1 x_1^S - w_2(1+t)x_2$$

Since x_1 is held fixed, we are left with only one first order necessary condition.

$$\pi_2 = pf_2(\overline{x}_1, x_2(p, w_2, w_1, t)) - w_2(1+t)$$

Note: With only one decision variable, our first and second order conditions change.

First order necessary condition: $\pi_2 = 0$

Second order sufficient condition: $f_{22} < 0$

At an interior solution, the input demand functions are implicitly defined by the FONC.

FONC: $pf_2(\overline{x}_1, x_2^{*S}) - w_2(1+t) \equiv 0$

Solving the FONC with respect to x_2 gives us $x_2^{*S}(p, w_1, w_2, t)$.

Differentiate π_2 *with respect to t:*

$$pf_{22}\frac{dx_2^{*S}}{dt} - w_2 = 0$$

$$\frac{dx_2^{*S}}{dt} = \frac{w_2}{pf_{22}} < 0$$

3 Points Total

1pt. New profit maximization set up and factor demand.

Note: Many of you said that f_1=0, which is true. However, the reason f_1=0, is because it is no longer a decision variable. It is a "given" in the short term. Thus, the set up of the profit maximization, the first and second order conditions are different. In order to get full points, you needed to recognize and show this in your answer.

1 pt Comparative Static

1pt Proving Le Chatelier's Principle

Compare long term response of factor to the tax to the short term response of factor two to the tax:
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Recall that $\frac{dx_2^L}{dt} = \frac{f_{11}w_2}{p(f_{11}f_{22} - f_{12}^2)} < 0$ from Part B and $\frac{dx_2^S}{dt} = \frac{w_2}{pf_{22}} < 0$

$$\frac{dx_2^L}{dt} - \frac{dx_2^S}{dt} = \frac{f_{11}w_2}{p(f_{11}f_{22} - f_{12}^2)} - \frac{w_2}{pf_{22}}$$

$$\frac{dx_2^L}{dt} - \frac{dx_2^S}{dt} = \frac{f_{22}(f_{11}w_2) - w_2(f_{11}f_{22} - f_{12}^2)}{p\ \ f_{22}(f_{11}f_{22} - f_{12}^2)}$$

$$\frac{dx_2^L}{dt} - \frac{dx_2^S}{dt} = \frac{w_2 f_{12}^2}{pf_{22}(f_{11}f_{22} - f_{12}^2)} < 0$$

Note: Since both comparative statics are negative, the "long run" comparative static must be MORE NEGATIVE than the "short run" comparative static. The difference between the two should thus be negative.

See Silberberg for additional information.

This proves that LeChatelier's Principle holds.

Question 2: Profit Maximization with Multiple Outputs **(Each part is worth 3.5 points for a total of 7 points)**

Assume that $C_1(y_1)$ and $C_2(y_2)$ are both twice continuously differentiable.

Part A:
The profit maximization problem is:

$$\underset{y_1,y_2}{Max}\left\{p(y_1+y_2)-C_1(y_1)-C_2(y_2): y_1, y_2 \geq 0\right\} = \underset{y_1}{Max}\left\{p(y_1)-C_1(y_1): y_1 \geq 0\right\} + \underset{y_2}{Max}\left\{p(y_2)-C_2(y_2): y_2 \geq 0\right\}$$

Which has for a solution $y_1^*(p), y_2^*(p),\ and\ \ y^*(p) = y_1^*(p) + y_2^*(p)$ if the FONC and the SOSC are both met.

The FONC are as follows:

$$\pi_1: \quad p - \frac{\partial C_1}{\partial y_1} = 0$$

$$\pi_2: \quad p - \frac{\partial C}{\partial y_2} = 0$$

In order for the solution to represent a global maximum, the SOSC must be met.

$$\pi_{yy} = H = \begin{bmatrix} -\frac{\partial^2 C_1}{\partial y_1^2} & 0 \\ 0 & -\frac{\partial^2 C_2}{\partial y_2^2} \end{bmatrix}$$

$$|H| = \frac{\partial^2 C_1}{\partial y_1^2} \cdot \frac{\partial^2 C_2}{\partial y_2^2} > 0, \quad -\frac{\partial^2 C_1}{\partial y_1^2} < 0, \ -\frac{\partial^2 C_2}{\partial y_2^2} < 0$$

Differentiate π_1 and π_2 with respect to p:

$$\frac{\partial^2 C_1}{\partial y_1^2}\frac{\partial y_1^*}{\partial p} = 1$$

$$\frac{\partial^2 C_2}{\partial y_2^2}\frac{\partial y_2^*}{\partial p} = 1$$

In matrix notation this becomes: $\begin{pmatrix} \frac{\partial^2 C_1}{\partial y_1^2} & 0 \\ 0 & \frac{\partial^2 C_2}{\partial y_2^2} \end{pmatrix}\begin{pmatrix} \frac{\partial y_1^*}{\partial p} \\ \frac{\partial y_2^*}{\partial p} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow -H\begin{pmatrix} \frac{\partial y_1^*}{\partial p} \\ \frac{\partial y_2^*}{\partial p} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solving for $\begin{pmatrix} \frac{\partial y_1^*}{\partial p} \\ \frac{\partial y_2^*}{\partial p} \end{pmatrix}$, we get the following: $\begin{pmatrix} \frac{\partial y_1^*}{\partial p} \\ \frac{\partial y_2^*}{\partial p} \end{pmatrix} = -H^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$-H^{-1} = \frac{\begin{bmatrix} \frac{\partial^2 C_2}{\partial y_2^2} & 0 \\ 0 & \frac{\partial^2 C_1}{\partial y_1^2} \end{bmatrix}}{\frac{\partial^2 C_2}{\partial y_2^2}\frac{\partial^2 C_1}{\partial y_1^2}} = \begin{bmatrix} \frac{1}{\frac{\partial^2 C_1}{\partial y_1^2}} & 0 \\ 0 & \frac{1}{\frac{\partial^2 C_2}{\partial y_2^2}} \end{bmatrix}.$$

Therefore, it follows that $\begin{pmatrix} \frac{\partial y_1^*}{\partial p} \\ \frac{\partial y_2^*}{\partial p} \end{pmatrix} = \begin{bmatrix} \frac{1}{\frac{\partial^2 C_1}{\partial y_1^2}} \\ \frac{1}{\frac{\partial^2 C_2}{\partial y_2^2}} \end{bmatrix}$ From the SOSC, we know that $\frac{\partial^2 C_1}{\partial y_1^2} > 0$ and $\frac{\partial^2 C_2}{\partial y_2^2} > 0$.

Therefore, $\frac{\partial y_1^*}{\partial p} = \frac{1}{\frac{\partial^2 C_1}{\partial y_1^2}} > 0$ and $\frac{\partial y_2^*}{\partial p} = \frac{1}{\frac{\partial^2 C_2}{\partial y_2^2}} > 0$. Since they are both greater than zeros, it implies that $\frac{\partial y^*}{\partial p} = \frac{\partial y_1^*}{\partial p} + \frac{\partial y_2^*}{\partial p} > 0$.

Part B:

The profit maximization problem is:

$$\underset{y_1, y_2}{Max}\left\{p(y_1+y_2) - ty_2 - C_1(y_1) - C_2(y_2) : y_1, y_2 \geq 0\right\} = \underset{y_1}{Max}\left\{p(y_1) - C_1(y_1) : y_1 \geq 0\right\} + \underset{y_2}{Max}\left\{p(y_2) - ty_2 - C_2(y_2) : y_2 \geq 0\right\}$$

Which has for a solution $y_1^*(p), y_2^*(p),$ *and* $y^*(p) = y_1^*(p) + y_2^*(p)$ if the FONC and the SOSC are both met.

The FONC are as follows:

$$\pi_1 : \quad p - \frac{\partial C_1}{\partial y_1} = 0$$

$$\pi_2 : \quad p - t - \frac{\partial C}{\partial y_2} = 0$$

In order for the solution to represent a global maximum, the SOSC must be met.

$$\pi_{yy} = H = \begin{bmatrix} -\frac{\partial^2 C_1}{\partial y_1^2} & 0 \\ 0 & -\frac{\partial^2 C_2}{\partial y_2^2} \end{bmatrix}$$

$$|H| = \frac{\partial^2 C_1}{\partial y_1^2} \cdot \frac{\partial^2 C_2}{\partial y_2^2} > 0, \quad -\frac{\partial^2 C_1}{\partial y_1^2} < 0, \ -\frac{\partial^2 C_2}{\partial y_2^2} < 0$$

Differentiate π_1 and π_2 with respect to t:

$$-\frac{\partial^2 C_1}{\partial y_1^2}\frac{\partial y_1^*}{\partial t}=0$$

$$-\frac{\partial^2 C_2}{\partial y_2^2}\frac{\partial y_2^*}{\partial t}=1$$

In matrix notation this becomes: $\begin{pmatrix} -\frac{\partial^2 C_1}{\partial y_1^2} & 0 \\ 0 & -\frac{\partial^2 C_2}{\partial y_2^2} \end{pmatrix}\begin{pmatrix} \frac{\partial y_1^*}{\partial t} \\ \frac{\partial y_2^*}{\partial t} \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to H\begin{pmatrix} \frac{\partial y_1^*}{\partial t} \\ \frac{\partial y_2^*}{\partial t} \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Solving for $\begin{pmatrix} \frac{\partial y_1^*}{\partial p} \\ \frac{\partial y_2^*}{\partial p} \end{pmatrix}$, we get the following: $\begin{pmatrix} \frac{\partial y_1^*}{\partial t} \\ \frac{\partial y_2^*}{\partial t} \end{pmatrix}=H^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$H^{-1}=\frac{\begin{bmatrix} -\frac{\partial^2 C_2}{\partial y_2^2} & 0 \\ 0 & -\frac{\partial^2 C_1}{\partial y_1^2} \end{bmatrix}}{\frac{\partial^2 C_2}{\partial y_2^2}\frac{\partial^2 C_1}{\partial y_1^2}}=\begin{bmatrix} -\frac{1}{\frac{\partial^2 C_1}{\partial y_1^2}} & 0 \\ 0 & -\frac{1}{\frac{\partial^2 C_2}{\partial y_2^2}} \end{bmatrix}.$$

Therefore, it follows that $\begin{pmatrix} \frac{\partial y_1^*}{\partial t} \\ \frac{\partial y_2^*}{\partial t} \end{pmatrix}=\begin{bmatrix} 0 \\ -\frac{1}{\frac{\partial^2 C_2}{\partial y_2^2}} \end{bmatrix}$ From the SOSC, we know that $\frac{\partial^2 C_2}{\partial y_2^2}>0$.

Therefore, $\frac{\partial y_2^*}{\partial t}=\frac{-1}{\frac{\partial^2 C_2}{\partial y_2^2}}<0$. Since they are both greater than zeros, it implies that $\frac{\partial y^*}{\partial t}=\frac{\partial y_1^*}{\partial t}+\frac{\partial y_2^*}{\partial t}<0$.

Question 3.

(1) The first-order condition for ~~fo~~ profit maximization is

$$\pi_x = Pf_x - w = 0$$

By implicit function theorem,

$$\frac{\partial x^*}{\partial w} = -(\pi_{xx})^{-1}\pi_{xw} = (\pi_{xx})^{-1} = \frac{1}{P}(f_{xx})^{-1}$$

since f_{xx} is negative definite, so is $(f_{xx})^{-1}$.

since $\frac{\partial x^*}{\partial w_l}$ is equal to the lth ~~entry~~ diagonal entry of $\frac{1}{P}(f_{xx})^{-1}$, we can conclude that $\frac{\partial x^*}{\partial w_l} < 0$.

(2) $MC = \frac{\partial C(w,y)}{\partial y}$.

$$\frac{\partial MC}{\partial w_1} = \frac{\partial^2 C(w,y)}{\partial y \partial w_1} = \frac{\partial^2 C(w,y)}{\partial w_1 \partial y} = \frac{\partial x_1^*}{\partial y} < 0.$$

(3) By profit maximization,

For $X = (a+b)^\alpha - \beta b^2$

$$\text{Max } \pi(a,b) = \max_{a,b} \{ P_x[(a+b)^\alpha - \beta b^2] - P_a a - P_b b \}.$$

FONC: $\frac{\partial \pi}{\partial a} = P_x \alpha (a+b)^{\alpha-1} - P_a = 0$ ①

$\frac{\partial \pi}{\partial b} = P_x[\alpha(a+b)^{\alpha-1} - 2\beta b] - P_b = 0$ ②

$\Rightarrow$ From ①, we get $\alpha(a+b)^{\alpha-1} = \frac{P_a}{P_x}$.

substitute into ②, we get

$$P_x\left(\frac{P_a}{P_x} - 2\beta b\right) = P_b$$

$\Rightarrow b^* = \frac{P_a - P_b}{2P_x\beta}$ $\quad b^*$ is greater than zero if $P_a > P_b$

substitute b^* into ①, we can get

$$a^* = \left(\frac{P_a}{P_x\alpha}\right)^{\frac{1}{\alpha-1}} - \frac{P_a - P_b}{2P_x\beta}$$

By substituting a^* and b^*, we can find

$$X^* = (a^*+b^*)^\alpha - \beta b^{*2} = \left(\frac{P_a}{P_x\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \beta\left(\frac{P_a - P_b}{2P_x\beta}\right)^2$$

check SOSC: $f_{aa} = \alpha(\alpha-1)(a+b)^{\alpha-2} < 0$

$$f_{bb} = \alpha(\alpha-1)(a+b)^{\alpha-2} - 2\beta < 0$$

$$f_{aa}f_{bb} - f_{ab}^2 = \alpha(\alpha-1)(a+b)^{\alpha-2}\left[\alpha(\alpha-1)(a+b)^{\alpha-2} - 2\beta\right]$$

$$-\left[\alpha(\alpha-1)(a+b)^{\alpha-2}\right]^2$$

$$= \alpha(1-\alpha)\, 2\beta\, (a+b)^{\alpha-2} > 0$$

The objective function is concave, (a^*, b^*) is the local max.

since $\dfrac{\partial b^*}{\partial x} = \dfrac{\partial b^*}{\partial P_x}\dfrac{\partial P_x}{\partial x} = \dfrac{-(P_a - P_b)2\beta}{4P_x^2\beta^2}\dfrac{dP_x}{dx} < 0$, assuming $\dfrac{dP_x}{dx} > 0$

$P_a > P_b$.

$$\frac{\partial a^*}{\partial x} = \frac{\partial a^*}{\partial P_x}\frac{\partial P_x}{\partial x} = \frac{(P_a - P_b)2\beta}{4P_x^2\beta^2}\frac{dP_x}{dx} > 0.$$

$\Rightarrow$ We can conclude that a is normal input and b is inferior input.

(3) By cost minimization,

$$\min_{a,b} \mathcal{L} = P_a a + P_b b + \lambda\left(\bar{X} - (a+b)^\alpha + \beta b^2\right).$$

(FONC): $\dfrac{\partial L}{\partial a} = 0 \Rightarrow P_a = \lambda\alpha(a+b)^{\alpha-1}$ ①

$\dfrac{\partial L}{\partial b} = 0 \Rightarrow P_b = \lambda\left[\alpha(a+b)^{\alpha-1} - 2\beta b\right].$ ②

$\dfrac{\partial L}{\partial \lambda} = 0 \Rightarrow \bar{X} = (a+b)^\alpha - \beta b^2.$ ③

check (SOSC): $\rightarrow$ Borded Hessian.

$$-\begin{vmatrix} -\lambda\alpha(\alpha-1)(a+b)^{\alpha-2} & -\lambda\alpha(\alpha-1)(a+b)^{\alpha-2} & -\alpha(a+b)^{\alpha-1} \\ -\lambda\alpha(\alpha-1)(a+b)^{\alpha-2} & -\lambda\alpha(\alpha-1)(a+b)^{\alpha-2} + 2\beta & -\alpha(a+b)^{\alpha-1} + 2\beta b \\ -\alpha(a+b)^{\alpha-1} & -\alpha(a+b)^{\alpha-1} + 2\beta b & 0 \end{vmatrix}_{a^*, b^*} > 0.$$

SOSC holds.

Divide ① into ② to yield.

$$\frac{P_b}{P_a} = 1 - \frac{2\beta b}{\alpha(a+b)^{\alpha-1}} \quad ④$$

solve ③ for a,

$$a = (\bar{X} + \beta b^2)^{\frac{1}{\alpha}} - b.$$

substitute into ④, we have

$$f(b, P_a, P_b, \bar{x}) = \frac{P_b}{P_a} - 1 + \frac{2\beta b}{\alpha(\bar{x}+\beta b^2)^{\frac{\alpha-1}{\alpha}}} = 0$$

$$\frac{\partial b}{\partial \bar{x}} = -\frac{f_{\bar{x}}}{f_b} = \frac{(\alpha-1)b}{\alpha(\bar{X}+\beta b^2) - 2\beta b^2(\alpha-1)} < 0.$$

$\Rightarrow$ b is inferior input.

Question 4.

(1) $h(x) = 3x_1 + 2x_2 = 12 \Rightarrow x_2 = 6 - \frac{3}{2}x_1$ ①

check CQ: $\frac{\partial h}{\partial x_1} = 3 \neq 0$ $\frac{\partial h}{\partial x_2} = 2 \neq 0$. CQ holds.

substitute ① into the objective function

$$\underset{x_1}{\text{Max}}\ x_1^3(6 - \tfrac{3}{2}x_1)$$

FONC: $3x_1^2(6-\frac{3}{2}x_1) - \frac{3}{2}x_1^3 = 0$

$$\Rightarrow \begin{cases} x_1^* = 3. \\ x_2^* = \frac{3}{2} \end{cases} \quad \text{or} \quad \begin{cases} x_1^* = 0 \\ x_2^* = 6 \end{cases}$$

(candidates are found!)

(2). check CQ as the above. (omitted here)

$$\mathcal{L} = x_1^3 x_2 + \lambda(12 - 3x_1 - 2x_2).$$

(FONC) $\frac{\partial L}{\partial x_1} = 3x_1^2 x_2 - 3\lambda = 0$ ②

$\frac{\partial L}{\partial x_2} = x_1^3 - 2\lambda = 0$ ③

$\frac{\partial L}{\partial \lambda} = 12 - 3x_1 - 2x_2 = 0.$ ④

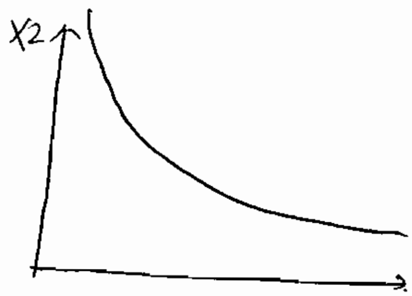
②/③, if $x_1 \neq 0$, then we have: $\frac{\cancel{3}x_2}{x_1} = \frac{\cancel{3}}{2} \Rightarrow x_2 = \frac{1}{2}x_1$ ⑤

Substitute ⑤ into ④, we have

$$12 - 3x_1 - 2\cdot\frac{1}{2}x_1 = 0 \Rightarrow \begin{cases} \lambda^* = \frac{27}{2} \\ x_1^* = 3 \\ x_2^* = \frac{3}{2}. \end{cases}$$

If $x_1 = 0$, then we have $\begin{cases} x_1^* = 0 \\ x_2^* = 6. \\ \lambda^* = 0 \end{cases}$ (candidates are found!)

(3). Level set: $x_1^3 x_2 = 32 \Rightarrow x_2 = g(x_1) = \frac{32}{x_1^3}$.



(4) In order to show the quasi-concave rigorously, you're encouraged to use borded Hessian.

$$f(x) = x_1^3 x_2$$

$$\frac{\partial f}{\partial x_1} = 3x_1^2 x_2$$

$$\frac{\partial f}{\partial x_2} = x_1^3$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_1} = 6x_1x_2$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 3x_1^2$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 3x_1^2$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_2} = 0$$

bordered ~~borded~~ hessian.

$$\begin{pmatrix} \cancel{3x_1^2x_2}\ 6x_1x_2 & 3x_1^2 & 3x_1^2x_2 \\ 3x_1^2 & 0 & x_1^3 \\ 3x_1^2x_2 & x_1^3 & 0 \end{pmatrix}$$

For $k = 1, 2$.

$$(-1)^{k=1}\det[C_1(x)] = (-1)\begin{vmatrix} 6x_1x_2 & 3x_1^2x_2 \\ 3x_1^2x_2 & 0 \end{vmatrix} = 9x_1^4x_2^2 > 0$$

$$(-1)^{k=2}\ \cancel{\det}\ \det[C_2(x)] = \begin{vmatrix} 6x_1x_2 & 3x_1^2 & 3x_1^2x_2 \\ 3x_1^2 & 0 & x_1^3 \\ 3x_1^2x_2 & x_1^3 & 0 \end{vmatrix} = 12x_1^7x_2 > 0.$$

We can conclude that $f(x)$ is quasi-concave.

(e). check SOSC in (a).

$$f_{11} = 36x_1 - 18x_1^2 = \begin{cases} 0, & \text{if } x_1 = 0 \\ -54, & \text{if } x_1 = 3 \end{cases}$$

Therefore when $x_1 = 0$, $f_{11} > 0$, violating the SOSC. We can rule out $(0, 6)$ as a solution.

However, when $x_1 = 3$, $f_{11} = -54 < 0$, the SOSC is satisfied. $(3, \frac{3}{2})$ is the ~~global interior~~ ~~tota~~ local interior max.

For (b)

$$\left[-\tfrac{3}{2} \quad 1\right] \begin{pmatrix} 6x_1x_2 & 3x_1^2 \\ 3x_1^2 & 0 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} \Bigg|_{x_1=0,\ x_2=6} = 0.$$

$(x_1, x_2) = (0, 6)$ is ruled out.

$$\left(-\tfrac{3}{2} \quad 1\right) \begin{pmatrix} 6x_1x_2 & 3x_1^2 \\ 3x_1^2 & 0 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} \Bigg|_{x_1=3,\ x_2=\frac{3}{2}}$$

$$= \left(-\tfrac{3}{2} \quad 1\right) \begin{pmatrix} 27 & 27 \\ 27 & 0 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} = -\frac{81}{4} < 0, \quad \text{SOSC holds here.}$$

$\Rightarrow$. $(3, \frac{3}{2})$ is the local interior max.

For special case, notice that $f(x_1, x_2)$ is quasi-concave. $h(x) = 12 - 3x_1 - 2x_2$ is linear, therefore both concave and convex, and quasi-concave. (and quasi-convex). We only need to check $f_1(x^*) \neq 0$, and $f_2(x^*) \neq 0$, and verify that $\lambda^* \geq 0$ for the FONC to be sufficient

$$f_1 = 3x_1^2x_2 \Big|_{x_1=3,\ x_2=\frac{3}{2}} = 3 \times \frac{3}{2} \times 3^2 \neq 0$$

$$f_2 = x_1^3 \Big|_{x_1=3} = 3^3 \neq 0.$$

$$\lambda^* = \frac{x_1^3}{2} \Big|_{x_1=3} = \frac{1}{2}\, 3^3 \geq 0$$

So the FONC is also sufficient in the case of $(3, \frac{3}{2})$.

In the case of $(0, 6)$. $f_1 = 0$, $f_2 = 0$, $\lambda^* = 0$, ruled out!

(4) λ^* is the value of relaxing the constraint by 1 unit. Hence λ^* is the additional output associated with an additional dollar spent purchasing input.

Comments:

1. For Q3 (1), you can also prove the result by using cost minimization, where a lot more work ~~is~~ needed due to the constrainted optimization.

2. I gave full credits to everyone for Q3 (2).

3. For Q3 (3), some of you used an alternative way:

From ① $\left(\frac{\partial \pi}{\partial a} = 0\right)$, we can get

$$\alpha (a+b)^{\alpha-1} = \frac{P_a}{P_x} \Rightarrow a+b = \left(\frac{\alpha P_x}{P_a}\right)^{\frac{1}{1-\alpha}}$$

$$\Rightarrow X = (a+b)^{\alpha} - \beta b^2 = \left(\frac{\alpha P_x}{P_a}\right)^{\frac{\alpha}{1-\alpha}} - \beta b^2.$$

Then. $\frac{\partial a}{\partial X} = \left(\frac{\partial X}{\partial a}\right)^{-1} = \left[\alpha (a+b)^{\alpha-1}\right]^{-1} = \left[\frac{P_a}{P_x}\right]^{-1} > 0.$

$$\frac{\partial b}{\partial X} = \left(\frac{\partial X}{\partial b}\right)^{-1} = \left[\alpha (a+b)^{\alpha-1} - 2\beta b\right]^{-1} = \left[\frac{P_a}{P_x} - \frac{P_a - P_b}{P_x}\right]^{-1}$$

$$= \cancel{\frac{P_a + P_b}{2P_x}} \left(\frac{P_b}{P_x}\right)^{-1} > 0.$$

This is not right.

One way to see it is, following the steps above, $\frac{\partial a}{\partial X} = \frac{P_x}{P_a}$

$$\frac{\partial b}{\partial X} = \frac{P_x}{P_b}.$$

In other words, $\begin{cases} P_a\, da = P_x\, dx \\ P_b\, db = P_x\, dx. \end{cases}$ $\Rightarrow$ It only holds if we don't allow for adjusting the combinations of a & b.

$\frac{\partial X}{\partial a} = \alpha (a+b)^{\alpha-1}$ is only the partial derivative.

Actually $X^* = X(a, b) = X(a, s(a))$. So you need to take the total differentiation to get the genuine $\frac{\partial X}{\partial a}$, which should be $\frac{\partial X}{\partial a} + \frac{\partial X}{\partial b} \cdot \frac{\partial b}{\partial a}$.

That's why in the suggested AK, I use $\frac{\partial a}{\partial P_x} \cdot \frac{\partial P_x}{\partial X}$ instead.

4. Question 4: you really need to show whether it's quasi-concave _ SOSC and so on.

5. To check Quasiconcavity, you've the following rules to follow.
(Please check class notes of Lecture 5)

Constructing $C_k(X) = \begin{pmatrix} f_{11} & \cdots & f_{1k} & f_1 \\ \vdots & & & \\ f_{k1} & \cdots & f_{kk} & f_k \\ f_1 & \cdots & f_k & 0 \end{pmatrix}_{(k+1)\times(k+1)}$ for $k=1,\cdots,n$

check: if $(-1)^k \det C_k(X) > 0$ for all x, then $f(x)$ is quasi-concave.

In the example of ~~k=2,~~ $n=2$, you need to check.

$$\begin{cases} \text{when } k=1, & (-1)^{k=1}\det C_1(X) = (-1)\begin{vmatrix} f_{11} & f_1 \\ f_1 & 0 \end{vmatrix} > 0 \quad \forall X \\ \text{when } k=2, & (-1)^{k=2}\det C_2(X) = (-1)^2\begin{vmatrix} f_{11} & f_{12} & f_1 \\ f_{21} & f_{22} & f_2 \\ f_1 & f_2 & 0 \end{vmatrix} > 0 \quad \forall X. \end{cases}$$

as the sufficient condition ~~to~~ for $f(x)$ to be quasi-concave.