Suggested solutions for homework #3:

Question 1:

1(a)

The firm's profit maximizing problem is:

$$\underset{x_1,x_2}{Max}\ \pi = pf(x_1,x_2) - w_1x_1 - (w_2{}^0 + kx_2)x_2 .$$

FONCs:

$\pi_{x_1} = pf_{x_1} - w_1 = 0;$

$\pi_{x_2} = pf_{x_2} - w_2{}^0 - 2kx_2 = 0.$

SOSC: The Hessian matrix of the profit function, $H = \pi_{\mathbf{xx}} = \begin{pmatrix} pf_{x_1x_1} & pf_{x_1x_2} \\ pf_{x_2x_1} & pf_{x_2x_2} - 2k \end{pmatrix}$, is negative definite, which implies

$pf_{x_1x_1} < 0 \ \Rightarrow\ f_{x_1x_1} < 0;$

$pf_{x_1x_1} \cdot (pf_{x_2x_2} - 2k) - (pf_{x_1x_2})^2 > 0 \ \Rightarrow pf_{x_1x_1} \cdot (pf_{x_2x_2} - 2k) > (pf_{x_1x_2})^2.$

Since $f_{x_1x_1} < 0$ and $(pf_{x_1x_2})^2 \geq 0$, $pf_{x_1x_1} \cdot (pf_{x_2x_2} - 2k) > (pf_{x_1x_2})^2$ also implies that $pf_{x_2x_2} - 2k < 0 \Rightarrow f_{x_2x_2} < \frac{2k}{p}$.

The "law of diminishing marginal product" still holds for x_1 as $f_{x_1x_1} < 0$. For x_2, the law will hold if $f_{x_2x_2} < 0$, but will not hold if $0 \leq f_{x_2x_2} < \frac{2k}{p}$ as both k and p are positive.

1(b)

Let $\boldsymbol{\theta} = [w_1, w_2{}^0, k]$, $\mathbf{x} = [x_1, x_2]$, then apply the implicit function theorem on the FONCs,

$$\mathbf{x}_{\boldsymbol{\theta}} = -(\pi_{\mathbf{xx}})^{-1}\pi_{\mathbf{x}\boldsymbol{\theta}} .$$

Write it out

$$\mathbf{x}_{\boldsymbol{\theta}} = \begin{pmatrix} \dfrac{\partial x_1}{\partial w_1} & \dfrac{\partial x_1}{\partial w_2{}^0} & \dfrac{\partial x_1}{\partial k} \\ \dfrac{\partial x_2}{\partial w_1} & \dfrac{\partial x_2}{\partial w_2{}^0} & \dfrac{\partial x_2}{\partial k} \end{pmatrix} = -\frac{1}{\det(\pi_{\mathbf{xx}})}\begin{pmatrix} pf_{x_2x_2} - 2k & -pf_{x_2x_1} \\ -pf_{x_1x_2} & pf_{x_1x_1} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -2x_2 \end{pmatrix}.$$

From the above equation, we can see that:

$$\frac{\partial x_1}{\partial w_2{}^0} = -\frac{1}{\det(\pi_{\mathbf{xx}})}\begin{pmatrix} pf_{x_2x_2} - 2k & -pf_{x_2x_1} \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\frac{pf_{x_2x_1}}{\det(\pi_{\mathbf{xx}})};$$

$$\frac{\partial x_2}{\partial w_1} = -\frac{1}{\det(\pi_{\mathbf{xx}})}\begin{pmatrix} -pf_{x_1x_2} & pf_{x_1x_1} \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\frac{pf_{x_1x_2}}{\det(\pi_{\mathbf{xx}})}.$$

Therefore $\dfrac{\partial x_1}{\partial w_2{}^0} = \dfrac{\partial x_2}{\partial w_1}$ as $f_{x_1x_2} = f_{x_2x_1}$ by Young's Theorem.

1(c)
With a tax t, the output price becomes $p-t$ (assume $p-t>0$). The firm's profit maximizing problem is:

$$\underset{x_1,x_2}{Max}\ \pi=(p-t)f(x_1,x_2)-w_1x_1-(w_2^{\,0}+kx_2)x_2$$

FONCs:

$$\pi_{x_1}=(p-t)f_{x_1}-w_1=0;$$

$$\pi_{x_2}=(p-t)f_{x_2}-w_2^{\,0}-2kx_2=0.$$

SOSC: The Hessian matrix becomes

$$H^t=\pi_{\mathbf{xx}}=\begin{pmatrix}(p-t)f_{x_1x_1} & (p-t)f_{x_1x_2}\\ (p-t)f_{x_2x_1} & (p-t)f_{x_2x_2}-2k\end{pmatrix}.$$

Again we need H^t to be negative definite for **x** to be differentiable with respect to the exogenous parameters, which gives the following conditions:

$$(p-t)f_{x_1x_1}<0\ \ \Rightarrow\ \ f_{x_1x_1}<0;$$

$$(p-t)f_{x_1x_1}\cdot\left((p-t)f_{x_2x_2}-2k\right)-((p-t)f_{x_1x_2})^2>0\text{ ; and}$$

$$(p-t)f_{x_2x_2}-2k<0.$$

Comparative Statics w.r.t. t:

$$\mathbf{x}_t=-\left(\pi_{\mathbf{xx}}\right)^{-1}\pi_{\mathbf{x}t}$$

Write it out

$$\begin{pmatrix}\dfrac{\partial x_1}{\partial t}\\ \dfrac{\partial x_2}{\partial t}\end{pmatrix}=-\frac{1}{\det(\pi_{\mathbf{xx}})}\begin{pmatrix}(p-t)f_{x_2x_2}-2k & -(p-t)f_{x_2x_1}\\ -(p-t)f_{x_1x_2} & (p-t)f_{x_1x_1}\end{pmatrix}\begin{pmatrix}-f_{x_1}\\ -f_{x_2}\end{pmatrix}$$

$$=\frac{1}{\det(\pi_{\mathbf{xx}})}\begin{pmatrix}\left((p-t)f_{x_2x_2}-2k\right)f_{x_1}-(p-t)f_{x_2x_1}f_{x_2}\\ (p-t)f_{x_1x_2}(f_{x_2}-f_{x_1})\end{pmatrix}.$$

The input use response to the tax can be either positive or negative, depending on the relative size of $f_{x_1}>0,\ f_{x_2}>0,\ f_{x_2x_2}<0,$ and the sign and size of $f_{x_1x_2}$. For example, when $f_{x_1}>(<)\ f_{x_2}$ and $f_{x_1x_2}>0$, $\dfrac{\partial x_2}{\partial t}<(>)\ 0$; when $f_{x_1}>(<)\ f_{x_2}$ and $f_{x_1x_2}<0$, $\dfrac{\partial x_2}{\partial t}>(<)\ 0$. The sign of $\dfrac{\partial x_1}{\partial t}$ is determined in a similar but more complicated way as above..

1(d)

With $p = ay + b$, the profit maximizing problem has a new set up:

$\underset{x_1, x_2}{Max}\ \pi^m = p(y) f(x_1, x_2) - w_1 x_1 - (w_2{}^0 + kx_2)x_2$,

where $p(y) = ay + b$ and $y = f(x_1, x_2)$.

FONCs:

$\pi^m_{x_1} = \frac{\partial p}{\partial x_1} f + pf_{x_1} - w_1 = 2af_{x_1} f + bf_{x_1} - w_1 = 0;$

$\pi^m_{x_2} = \frac{\partial p}{\partial x_2} f + pf_{x_2} - w_2{}^0 - 2kx_2 = 2af_{x_2} f + bf_{x2} - w_2{}^0 - 2kx_2 = 0.$

SOSCs:

The new Hessian matrix of the profit function,

$H^p = \pi^m_{\mathbf{xx}} = \begin{pmatrix} (2af + b) f_{x_1x_1} + 2a(f_{x_1})^2 & (2af + b) f_{x_1x_2} + 2af_{x_1} f_{x_2} \\ (2af + b) f_{x_1x_2} + 2af_{x_1} f_{x_2} & (2af + b) f_{x_2x_2} + 2a(f_{x_2})^2 - 2k \end{pmatrix}$, is negative definite for $\mathbf{x}$ to be differentiable with respect to the exogenous parameters, which implies

$(2af + b) f_{x_1x_1} + 2a(f_{x_1})^2 < 0$;

$\det(\pi^m_{\mathbf{xx}}) = [(2af + b) f_{x_1x_1} + 2a(f_{x_1})^2] \cdot [(2af + b) f_{x_2x_2} + 2a(f_{x_2})^2 - 2k] - [(2af + b) f_{x_1x_2} + 2af_{x_1} f_{x_2}]^2 > 0$; and

$(2af + b) f_{x_2x_2} + 2a(f_{x_2})^2 - 2k < 0.$

Comparative Statics:

Given the SOSCs hold, we can do the comparative statics as following

$\mathbf{x}_{w_1} = -\left(\pi^m_{\mathbf{xx}}\right)^{-1} \pi^m_{\mathbf{x}w_1}$.

Write it out:

$$\begin{pmatrix} \dfrac{\partial x_1}{\partial w_1} \\ \dfrac{\partial x_2}{\partial w_1} \end{pmatrix} = -\frac{1}{\det(\pi^m_{\mathbf{xx}})} \begin{pmatrix} (2af + b) f_{x_2x_2} + 2a(f_{x_2})^2 - 2k & -(2af + b) f_{x_1x_2} - 2af_{x_1} f_{x_2} \\ -(2af + b) f_{x_1x_2} - 2af_{x_1} f_{x_2} & (2af + b) f_{x_1x_1} + 2a(f_{x_1})^2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\det(\pi^m_{\mathbf{xx}})} \begin{pmatrix} (2af + b) f_{x_2x_2} + 2a(f_{x_2})^2 - 2k \\ -(2af + b) f_{x_1x_2} - 2af_{x_1} f_{x_2} \end{pmatrix}.$$

From the SOSCs, we know $\det(\pi_{\mathbf{xx}}) > 0$, and $(2af + b) f_{x_2x_2} + 2a(f_{x_2})^2 - 2k < 0$, thus $\frac{\partial x_1}{\partial w_1} < 0$. The sign of $\frac{\partial x_2}{\partial w_1}$ can be positive or negative, depending on the relative size of *a*, *b*, f_{x_1}, f_{x_2}, $f(\mathbf{x}^*)$, and the sign and size of $f_{x_1x_2}$.

1(e)

When x_2 is fixed, say $x_2 = \bar{x}_2$, the profit maximizing problem becomes:

$$\underset{x_1}{Max}\ \pi = pf(x_1, \bar{x}_2) - w_1 x_1 - (w_2^0 + k\bar{x}_2)\bar{x}_2 .$$

FONC:

$$pf_{x_1} - w_1 = 0$$

SOSC:

$$pf_{x_1x_1} < 0 \ \Rightarrow\ f_{x_1x_1} < 0 .$$

"x_1 is downward-sloping in w_1" means $\frac{\partial x_1}{\partial w_1} < 0$. Apply the implicit function theorem to the FONC, we have

$$\frac{\partial x_1}{\partial w_1} = -\left(\pi_{x_1x_1}\right)^{-1}\pi_{x_1w_1} = -\left(pf_{x_1x_1}\right)^{-1}(-1) = \frac{1}{pf_{x_1x_1}} < 0 .$$

Therefore the input demand function x_1 is downward-sloping in w_1.

<u>Question 2:</u>

<u>2(a):</u>

Min $\ f(x_1, x_2) = 2x_1 + x_2$

s.t. $2 - x_1x_2 = 0.$

Check CQ conditions:

Let $h = 2 - x_1x_2$, then $h_{x_2} = -x_1 \neq 0$ if $x_1 \neq 0$. So as long as x_1 does not equal to 0, the CQ condition holds.

1) Do it by substitution:

The constraint gives $x_2 = \frac{2}{x_1}$, plug it in the object function:

$$f = f(x_1) = 2x_1 + \frac{2}{x_1}$$

FONC: $f_{x_1} = 2 - \frac{2}{x_1^2} = 0$

This gives two candidates: $x_1^* = -1$ or $x_1^* = 1$. Both candidates satisfy the CQ condition.

SOSC: $\quad f_{x_1x_1} = \frac{d}{dx_1}\left(2 - \frac{2}{x_1^2}\right) = -2(-2x_1^{-3}) = \frac{4}{x_1^3} > 0$

Since $f_{x_1x_1} > 0$ when $x_1^* = 1$, $f_{x_1x_1} < 0$ when $x_1^* = -1$, we know the solution is $x_1^* = 1$ for this minimization problem.

2). Do it by Lagrangian:
Set up the Lagrange as
$L = L(x_1, x_2, \lambda) = 2x_1 + x_2 + \lambda(2 - x_1 x_2)$
FONCs:

$$\frac{\partial L}{\partial x_1} = 2 - \lambda x_2 = 0; \qquad (1)$$

$$\frac{\partial L}{\partial x_2} = 1 - \lambda x_1 = 0; \qquad (2)$$

$$\frac{\partial L}{\partial x_1} = 2 - x_1 x_2 = 0 \qquad (3)$$

The first two equations gives

$$\lambda = \frac{2}{x_2};\ \lambda = \frac{1}{x_1} \quad \Rightarrow \quad \frac{2}{x_2} = \frac{1}{x_1} \quad \Rightarrow x_2 = 2x_1$$

Plug it into the third equation:
$2 - x_1 \bullet 2x_1 = 0 \quad \Rightarrow x_1^* = -1$ or $x_1^* = 1$. Both candidates satisfy the CQ condition.
If $x_1^* = -1$, $\lambda^* = -1$, which violates the condition that λ^* has to be positive. So $x_1^* = 1$, $x_2^* = 2$, $\lambda^* = 1$ is the only candidate if we take account of the restriction on λ^*.

But we still need the SOC to verify that this candidate is indeed the optimal solution *(you can also use the quasi-convexity to establish that FONCs are also sufficient)*:. For this case, we only need to show that the determinant of the bordered Hessian is negative evaluated at the optimal value. The bordered Hessian is

$$H^b = \begin{pmatrix} L_{x_1x_1} & L_{x_1x_2} & h_{x_1} \\ L_{x_2x_1} & L_{x_2x_2} & h_{x_2} \\ h_{x_1} & h_{x_2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\lambda & -\lambda x_2 \\ -\lambda & 0 & -\lambda x_1 \\ -\lambda x_2 & -\lambda x_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -2 \\ -1 & 0 & -1 \\ -2 & -1 & 0 \end{pmatrix}$$

when evaluated at the candidate point ($x_1^* = 1$, $x_2^* = 2$, $\lambda^* = 1$).

$$\det(H^b) = -(-1)\det\begin{pmatrix} -1 & -1 \\ -2 & 0 \end{pmatrix} + (-2)\det\begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} = -4 < 0$$

Which does satisfy the second order condition, therefore $x_1^* = 1$, $x_2^* = 2$, $\lambda^* = 1$ is the optimal interior solution.

__2(b):__

$$Max \quad f(x_1,\ x_2) = x_1 x_2$$
$$s.t. \quad 4 - (2x_1 + x_2) = 0;$$

Check CQ conditions:

Let $h = 4-(2x_1 + x_2)$, then $h_{x_2} = -1 \neq 0$. So the CQ condition holds for all x_1 and x_2.

1) by substitution:
The constraint gives $x_2 = 4-2x_1$, plug it in the object function:
$f = f(x_1) = x_1(4-2x_1)$
FONC: $f_{x_1} = 4-4x_1 = 0$
This gives: $x_1^* = 1$.
SOSC: $f_{x_1x_1} = -4 < 0$
So $x_1^* = 1$ is the optimal solution for this minimization problem.

2). by Lagrangian:
Set up the Lagrange as
$L = L(x_1, x_2, \lambda) = x_1x_2 + \lambda(4-2x_1 - x_2)$
FONCs:

$$\frac{\partial L}{\partial x_1} = x_2 - 2\lambda = 0;$$

$$\frac{\partial L}{\partial x_2} = x_1 - \lambda = 0;$$

$$\frac{\partial L}{\partial x_1} = 4-2x_1 - x_2 = 0$$

Solve it, we get $x_1^* = 1$, $x_2^* = 2$, $\lambda^* = 1$.

SOSC *(you can also use the quasi-concavity to establish that FONCs are also sufficient)*:
We still need the SOC to verify that this candidate is indeed the optimal solution. For this case, we need to show that the determinant of the bordered Hessian is positive when evaluated at the optimal value. The bordered Hessian is

$$H^b = \begin{pmatrix} L_{x_1x_1} & L_{x_1x_2} & h_{x_1} \\ L_{x_2x_1} & L_{x_2x_2} & h_{x_2} \\ h_{x_1} & h_{x_2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & -1 \\ -2 & -1 & 0 \end{pmatrix}$$

$$\det(H^b) = -1 \bullet \det\begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix} + (-2)\det\begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} = 4 > 0$$

Which does satisfy the second order condition, therefore $x_1^* = 1$, $x_2^* = 2$, $\lambda^* = 1$ is the optimal interior solution.

Discuss:
The numerical value of the Lagrange Multiplier λ^* from the two problems are the same, but they have different economic meanings. In part a, the exogenous value in the constraint is related to the target production level, so λ^* means the shadow value (or cost) of relaxing the production target by one more unit. In part b, the exogenous value in the

constraint is related to the target total cost level (e.g. an expenditure budget), so λ^* means the shadow value of relaxing the budget by one more unit. (YOU ARE FINE IF YOU GET HERE)

(Some additional comments): In the equilibrium, the shadow cost of the output (λ^* in terms of \$/output) should be equal to the inverse of the shadow value of the additional budget (λ^* in terms of output/\$). In this case, since the inverse of 1 is still 1, they happen to be equal to each other. Moreover, we can see that the solutions from the two parts above are exactly the same. And the value of the object function in part a) is 4, which is the same as the exogenous value in the constraint in part b. The value of the object function in part b is 2, which is the same as the exogenous value in the constraint in part a. This gives us an example of the equivalence (or duality) between firm's cost minimization problem given a target production level and production maximization problem given an expenditure budget.

Question 3:

The problem set up is:

$$\begin{aligned} &\underset{x_1,x_2}{Max} \ \ f(x_1,x_2,\alpha) \\ &s.t. \ \ g(x_1,x_2)=k \end{aligned}.$$

The Lagrangian:

$$L = f(x_1,x_2,\alpha)+\lambda(k-g(x_1,x_2)).$$

3(a): prove $\frac{\partial^2 f}{\partial x_1 \partial \alpha}\cdot\frac{\partial x_1^*}{\partial k}+\frac{\partial^2 f}{\partial x_2 \partial \alpha}\cdot\frac{\partial x_2^*}{\partial k}=\frac{\partial \lambda^*}{\partial \alpha}.$

From the equation we are going to prove, we know that we need to do a comparative statics of x_1^*, x_1^* and λ^* with respect to k and α.

In order to do the comparative statics, we need the bordered hessian matrix to be invertible. So

$$H^b=\begin{pmatrix} f_{x_1x_1}-\lambda g_{x_1x_1} & f_{x_1x_2}-\lambda g_{x_1x_2} & -g_{x_1} \\ f_{x_1x_2}-\lambda g_{x_1x_2} & f_{x_2x_2}-\lambda g_{x_2x_2} & -g_{x_2} \\ -g_{x_1} & -g_{x_2} & 0 \end{pmatrix}\equiv\begin{pmatrix} L_{11} & L_{12} & -g_1 \\ L_{12} & L_{22} & -g_2 \\ -g_1 & -g_2 & 0 \end{pmatrix}.$$

The last identity is used for notation convenience. The condition is that $\det(H^b)\neq 0$.

We know the FONCs are: (I omit the star in the following notations)

$$L_{x_1} = f_{x_1}(x_1,x_2,\alpha)-\lambda g_{x_1}(x_1,x_2)=0;$$

$$L_{x_1} = f_{x_2}(x_1,x_2,\alpha)-\lambda g_{x_2}(x_1,x_2)=0;$$

$$L_\lambda = k-g(x_1,x_2)=0.$$

These equations implicitly defined x_1^*, x_1^* and λ^* as functions of k and α . Apply implicit function theorem, we know the formula for comparative statics is:
$x_w = -\left(L_{xx}\right)^{-1} L_{xw} \equiv -\left(H^b\right)^{-1} L_{xw}$,
where $x = [x_1, x_2, \lambda]^T$, $w = [\alpha, k]^T$ here. Write it out:

$$\begin{pmatrix} \frac{\partial x_1}{\partial \alpha} & \frac{\partial x_1}{\partial k} \\ \frac{\partial x_2}{\partial \alpha} & \frac{\partial x_2}{\partial k} \\ \frac{\partial \lambda}{\partial \alpha} & \frac{\partial \lambda}{\partial k} \end{pmatrix} = -\frac{1}{\det(H^b)} \begin{pmatrix} \begin{vmatrix} L_{22} & -g_2 \\ -g_2 & 0 \end{vmatrix} & -\begin{vmatrix} L_{12} & -g_1 \\ -g_2 & 0 \end{vmatrix} & \begin{vmatrix} L_{12} & -g_1 \\ L_{22} & -g_2 \end{vmatrix} \\ -\begin{vmatrix} L_{12} & -g_2 \\ -g_1 & 0 \end{vmatrix} & \begin{vmatrix} L_{11} & -g_1 \\ -g_1 & 0 \end{vmatrix} & -\begin{vmatrix} L_{11} & -g_1 \\ L_{12} & -g_2 \end{vmatrix} \\ \begin{vmatrix} L_{12} & L_{22} \\ -g_1 & -g_2 \end{vmatrix} & -\begin{vmatrix} L_{11} & L_{12} \\ -g_1 & -g_2 \end{vmatrix} & \begin{vmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{vmatrix} \end{pmatrix} \begin{pmatrix} f_{x_1\alpha} & 0 \\ f_{x_2\alpha} & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore

$$\frac{\partial \lambda}{\partial \alpha} = -\frac{1}{\det(H^b)} \left(\begin{vmatrix} L_{12} & L_{22} \\ -g_1 & -g_2 \end{vmatrix} \quad -\begin{vmatrix} L_{11} & L_{12} \\ -g_1 & -g_2 \end{vmatrix} \quad \begin{vmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{vmatrix} \right) \begin{pmatrix} f_{x_1\alpha} \\ f_{x_2\alpha} \\ 0 \end{pmatrix}$$

$$= -\frac{1}{\det(H^b)} \left(\begin{vmatrix} L_{12} & L_{22} \\ -g_1 & -g_2 \end{vmatrix} f_{x_1\alpha} - \begin{vmatrix} L_{11} & L_{12} \\ -g_1 & -g_2 \end{vmatrix} f_{x_2\alpha} \right)$$

$$= -\frac{1}{\det(H^b)} \begin{vmatrix} L_{12} & L_{22} \\ -g_1 & -g_2 \end{vmatrix} f_{x_1\alpha} + \frac{1}{\det(H^b)} \begin{vmatrix} L_{11} & L_{12} \\ -g_1 & -g_2 \end{vmatrix} f_{x_2\alpha};$$

$$\frac{\partial x_1}{\partial k} = -\frac{1}{\det(H^b)} \left(\begin{vmatrix} L_{22} & -g_2 \\ -g_2 & 0 \end{vmatrix} \quad -\begin{vmatrix} L_{12} & -g_1 \\ -g_2 & 0 \end{vmatrix} \quad \begin{vmatrix} L_{12} & -g_1 \\ L_{22} & -g_2 \end{vmatrix} \right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= -\frac{1}{\det(H^b)} \begin{vmatrix} L_{12} & -g_1 \\ L_{22} & -g_2 \end{vmatrix} = -\frac{1}{\det(H^b)} \begin{vmatrix} L_{12} & L_{22} \\ -g_1 & -g_2 \end{vmatrix}.$$

Similarly,

$$\frac{\partial x_2}{\partial k} = -\frac{1}{\det(H^b)} \left(-\begin{vmatrix} L_{12} & -g_2 \\ -g_1 & 0 \end{vmatrix} \quad \begin{vmatrix} L_{11} & -g_1 \\ -g_1 & 0 \end{vmatrix} \quad -\begin{vmatrix} L_{11} & -g_1 \\ L_{12} & -g_2 \end{vmatrix} \right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\det(H^b)} \begin{vmatrix} L_{11} & -g_1 \\ L_{12} & -g_2 \end{vmatrix} = \frac{1}{\det(H^b)} \begin{vmatrix} L_{11} & L_{12} \\ -g_1 & -g_2 \end{vmatrix}.$$

From the above results, we can see that

$$\frac{\partial \lambda}{\partial \alpha} = \frac{\partial x_1}{\partial k} f_{x_1\alpha} + \frac{\partial x_2}{\partial k} f_{x_2\alpha}.$$

Be aware that all the x_1, x_2 and λ in the comparative statics should be x_1^*, x_2^* and λ^*, i.e., they are all at the optimal level.

<u>3(b)</u> In order to make $\frac{\partial x_1^*}{\partial k} = \frac{\partial \lambda^*}{\partial \alpha}$, two conditions need to be satisfied:
$\frac{\partial^2 f}{\partial x_1 \partial \alpha} = 1$; and $\frac{\partial^2 f}{\partial x_2 \partial \alpha} = 0$ or $\frac{\partial x_2^*}{\partial k} = 0$. These conditions are satisfied if
$f(x_1, x_2; \alpha) = g(x_1, x_2) + \alpha x_1 + C$ where C is any constant.