Suggested solutions to homework 2

1(a)

One sufficient condition that the production function is strictly concave is if the Hessian matrix is negative definite. The Hessian matrix of the production function is

$$\mathbf{H} = \begin{bmatrix} Y_{LL} & Y_{LK} \\ Y_{KL} & Y_{KK} \end{bmatrix} = \begin{bmatrix} \alpha(\alpha-1)AL^{\alpha-2}K^{\beta} & \alpha\beta AL^{\alpha-1}K^{\beta-1} \\ \alpha\beta AL^{\alpha-1}K^{\beta-1} & \beta(\beta-1)AL^{\alpha}K^{\beta-2} \end{bmatrix}.$$

The conditions for **H** to be negative definite is

$$\det(C^1) = \alpha(\alpha-1)AL^{\alpha-2}K^{\beta} < 0;$$

$$\det(C^2) = \det(\mathbf{H}) = \alpha(\alpha-1)AL^{\alpha-2}K^{\beta}\beta(\beta-1)AL^{\alpha}K^{\beta-2} - (\alpha\beta AL^{\alpha-1}K^{\beta-1})^2$$
$$= \alpha\beta(1-\alpha-\beta)A^2L^{2\alpha-2}K^{2\beta-2} > 0;$$

For these two inequality to hold, we need $\alpha < 1$, and $\alpha+\beta<1$. As both α and β are positive, the conditions are rewritten as $\alpha+\beta < 1$, $\alpha > 0$, and $\beta > 0$.

1(b)

The firm would like to

$$\underset{K,L}{Max}\ \pi = pY - wL - rK,$$

where $Y = AL^{\alpha}K^{\beta}$ by technology efficiency implied by profit maximization.

The FONCs are:

$$Y_L = p\cdot\alpha AL^{\alpha-1}K^{\beta} - w = 0 \quad (1)$$
$$Y_K = p\cdot\beta AL^{\alpha}K^{\beta-1} - r = 0 \quad (2)$$

Solve them to get the optimal K^* and L^* in terms of all other parameters. We have:

$$L^* = \left(\frac{p\alpha A}{w}\right)^{\frac{1}{1\text{-}\alpha-\beta}}\left(\frac{\beta w}{\alpha r}\right)^{\frac{\beta}{1\text{-}\alpha-\beta}}$$

$$K^* = \left(\frac{p\alpha A}{w}\right)^{\frac{1}{1\text{-}\alpha-\beta}}\left(\frac{\beta w}{\alpha r}\right)^{\frac{1-\alpha}{1\text{-}\alpha-\beta}}$$

1(c)

Given the profit function $\pi = pY - wL - rK = pY - \mathbf{wx}$, we can derive

$\pi_{\mathbf{xx}} = p\cdot\dfrac{\partial^2 Y}{\partial(K,L)^2} = p\mathbf{H}$. We have shown in part (a) that **H** is negative definite, and by assumption $p > 0$, therefore $\pi_{\mathbf{xx}}$ is also negative definite, which means the SOSC is satisfied.

We now construct the FONC identity as $\frac{\partial \pi(K^*(p,r,w), L^*(p,r,w))}{\partial(K,L)} \equiv 0$. Since the SOSC is satisfied (see the discussion above), $\pi_{\mathbf{xx}}$ is invertible, thus we can use the implicit function theorem to obtain the comparative statics of K^* and L^* with respect to the changes in input prices r and w:

$$\frac{\partial(K^*(p,r,w), L^*(p,r,w))}{\partial(r,w)} = \begin{bmatrix} \frac{\partial K}{\partial r} & \frac{\partial L}{\partial r} \\ \frac{\partial K}{\partial w} & \frac{\partial L}{\partial w} \end{bmatrix} = -(\frac{\partial^2 \pi}{\partial \mathbf{x}^2})^{-1} \frac{\partial^2 \pi}{\partial \mathbf{xw}} = -(\pi_{\mathbf{xx}})^{-1}(-\mathbf{I}) = (\pi_{\mathbf{xx}})^{-1},$$

where $\mathbf{x} = (K, L)$, $\mathbf{w} = (r, w)$.

We derived earlier that $\pi_{\mathbf{xx}} = p \cdot \frac{\partial^2 Y}{\partial(K,L)^2} = p\mathbf{H}$, and in part (a): $\mathbf{H} =$

$$\begin{bmatrix} \alpha(\alpha-1)AL^{\alpha-2}K^{\beta} & \alpha\beta AL^{\alpha-1}K^{\beta-1} \\ \alpha\beta AL^{\alpha-1}K^{\beta-1} & \beta(\beta-1)AL^{\alpha}K^{\beta-2} \end{bmatrix}.$$ We can then obtain the inverse matrix

$$\mathbf{H}^{-1} = \frac{1}{\det(\mathbf{H})} \begin{bmatrix} \beta(\beta-1)AL^{\alpha}K^{\beta-2} & -\alpha\beta AL^{\alpha-1}K^{\beta-1} \\ -\alpha\beta AL^{\alpha-1}K^{\beta-1} & \alpha(\alpha-1)AL^{\alpha-2}K^{\beta} \end{bmatrix},$$ where det($\mathbf{H}$) is obtained in part (a).

Thus the comparative static results are:

$\frac{\partial K}{\partial w} = \frac{-\alpha\beta AL^{\alpha-1}K^{\beta-1}}{p\det(\mathbf{H})} < 0$, as we have shown in part (a) that det($\mathbf{H}$) > 0 and α, β, A are all positive. Similarly $\frac{\partial L}{\partial r} = \frac{-\alpha\beta AL^{\alpha-1}K^{\beta-1}}{p\det(\mathbf{H})} < 0$, and $\frac{\partial K}{\partial w} = \frac{\partial L}{\partial r}$.

<u>1(d)</u>

Note that a tax t/unit is virtually a reduction in output price p, i.e. $(p - t)$, thus

$$\frac{\partial Y^*}{\partial t} = \frac{\partial Y^*}{\partial p}\frac{\partial(p-t)}{\partial t} = -\frac{\partial Y^*}{\partial p}.$$

We have $\frac{\partial Y^*}{\partial p} = A\alpha(L^*)^{\alpha-1}(K^*)^{\beta}\frac{\partial L^*}{\partial p} + A\beta(L^*)^{\alpha}(K^*)^{\beta-1}\frac{\partial K^*}{\partial p}$.

From solutions to part (b), $\frac{\partial L^*}{\partial p} = \frac{1}{1-\alpha-\beta}(\frac{p\alpha A}{w})^{\frac{\alpha+\beta}{1-\alpha-\beta}}(\frac{\beta w}{\alpha r})^{\frac{\beta}{1-\alpha-\beta}} > 0$, as we have shown in part (a) that $\alpha + \beta < 1$, and α, β, A are all positive.

Similarly $\frac{\partial K^*}{\partial p} = \frac{1}{1-\alpha-\beta}(\frac{p\alpha A}{w})^{\frac{\alpha+\beta}{1-\alpha-\beta}}(\frac{\beta w}{\alpha r})^{\frac{1-\alpha}{1-\alpha-\beta}} > 0$.

Therefore we can conclude that $\frac{\partial Y^*}{\partial p} = A\alpha(L^*)^{\alpha-1}(K^*)^{\beta}\frac{\partial L^*}{\partial p} + A\beta(L^*)^{\alpha}(K^*)^{\beta-1}\frac{\partial K^*}{\partial p} > 0$,

which leads to $\frac{\partial Y^*}{\partial t} = -\frac{\partial Y^*}{\partial p} < 0$: the firm will respond to the imposed tax by cutting production.

1(e)

If the firm cannot adjust K in the short run, the profit maximizing problem is reduced to

$$\underset{L}{Max}\ \pi = pAL^{\alpha}(\overline{K})^{\beta} - wL - r\overline{K}.$$

The short run solution is given by the following FONC:

$\frac{\partial \pi}{\partial L} = pA\alpha L^{\alpha-1}(\overline{K})^{\beta} - w = 0$, ➔ $\tilde{L} = (\frac{\alpha AP}{w}\overline{K}^{\beta})^{\frac{1}{1-\alpha}}$.

Indeed if $\overline{K} = K^*$, ➔ $\tilde{L} = (\frac{\alpha AP}{w}(K^*)^{\beta})^{\frac{1}{1-\alpha}} = L^*$.

The short run response to the tax is now $\frac{\partial \tilde{Y}}{\partial t} = -\frac{\partial \tilde{Y}}{\partial p} = A\alpha(L)^{\alpha-1}(\overline{K})^{\beta}\frac{\partial L}{\partial p}$.

For comparison purpose, the short run and long run response should be starting from the original profit maximizing production points and we then investigate how the optimal production level responds to the newly imposed tax policy, therefore initially $\overline{K} = K^*$, we then have $\tilde{L} = L^*$.

Now the long run and short run responses are:

$$\frac{\partial Y^*}{\partial t} = -\frac{\partial Y^*}{\partial p} = -[A\alpha(L^*)^{\alpha-1}(K^*)^{\beta}\frac{\partial L^*}{\partial p} + A\beta(L^*)^{\alpha}(K^*)^{\beta-1}\frac{\partial K^*}{\partial p}],$$

$$\frac{\partial \tilde{Y}}{\partial t} = -\frac{\partial \tilde{Y}}{\partial p} = -A\alpha(\tilde{L})^{\alpha-1}(\overline{K})^{\beta}\frac{\partial \tilde{L}}{\partial p} = -A\alpha(L^*)^{\alpha-1}(K^*)^{\beta}\frac{\partial L^*}{\partial p}.$$

We obtain $\frac{\partial Y^*}{\partial t} - \frac{\partial \tilde{Y}}{\partial t} = -A\beta(L^*)^{\alpha}(K^*)^{\beta-1}\frac{\partial K^*}{\partial p} < 0$, i.e., the long run production level decreases more than the short run production level, or more intuitively

$|\frac{\partial Y^*}{\partial t}| - |\frac{\partial \tilde{Y}}{\partial t}| = A\beta(L^*)^{\alpha}(K^*)^{\beta-1}\frac{\partial K^*}{\partial p} > 0$, i.e. the long run decisions are more responsive than the short run decisions.

YOU MAY ALSO SOLVE THE PROBLEMS USING TRADITIONAL DERIVATIVE METHODS. SEE THE FOLLOWING FOR SOME ALTERNATIVE SOLUTIONS.

HOWEVER, IT IS IMPORTANT THAT YOU CAN DO THESE COMPARATIVE STATICS IN MATRIX NOTATIONS, WHICH IS THE GENERAL APPROACH EVEN WITH COMPLICATED FUNCTION FORMS WITH MANY INPUTS.

Problem 1: Analysis on Cobb-Douglas Production Function

a) Conditions for Concavity

For a function to be strictly concave, its Hessian matrix needs to be negative definite. So we need to get the Hessian matrix first. As $Y = AL^{\alpha}K^{\beta}$, the first derivatives of Y w.r.t L and K are

$$\frac{\partial Y}{\partial L} = \alpha A L^{\alpha-1}K^{\beta}$$
$$\frac{\partial Y}{\partial K} = \beta A L^{\alpha}K^{\beta-1}$$

Its second derivatives are

$$\frac{\partial^2 Y}{\partial L^2} = \alpha(\alpha-1)AL^{\alpha-2}K^{\beta}$$
$$\frac{\partial^2 Y}{\partial K^2} = \beta(\beta-1)AL^{\alpha}K^{\beta-2}$$
$$\frac{\partial^2 Y}{\partial L \partial K} = \frac{\partial^2 Y}{\partial K \partial L} = \alpha\beta AL^{\alpha-1}K^{\beta-1}$$

Therefore the Hessian matrix H is

$$H = \begin{bmatrix} \alpha(\alpha-1)AL^{\alpha-2}K^{\beta} & \alpha\beta AL^{\alpha-1}K^{\beta-1} \\ \alpha\beta AL^{\alpha-1}K^{\beta-1} & \beta(\beta-1)AL^{\alpha}K^{\beta-2} \end{bmatrix}$$

For H to be negative definite, these two conditions need to be hold:

$$det(H_1) < 0$$
$$det(H_2) > 0$$

where

$$\begin{aligned} det(H_1) &= \alpha(\alpha-1)AL^{\alpha-2}K^{\beta} \\ det(H_2) &= \alpha(\alpha-1)AL^{\alpha-2}K^{\beta} \cdot \beta(\beta-1)AL^{\alpha}K^{\beta-2} - (\alpha\beta AL^{\alpha-1}K^{\beta-1})^2 \\ &= \left[\alpha\beta(\alpha-1)(\beta-1) - \alpha^2\beta^2\right] A^2L^{2\alpha-2}K^{2\beta-2} \\ &= \left[\alpha\beta(\alpha\beta-\alpha-\beta+1) - \alpha^2\beta^2\right] A^2L^{2\alpha-2}K^{2\beta-2} \\ &= \alpha\beta(1-\alpha-\beta)\, A^2L^{2\alpha-2}K^{2\beta-2} \end{aligned}$$

This gives us

$$\alpha < 1; \qquad \alpha + \beta < 1$$

As $\beta > 0$ is given, these conditions reduce to

$$\alpha + \beta < 1$$

b) Profit-maximizing Input and Supply Functions

The profit-maximizing problem for the competitive firm is

$$\max_{L,K} \quad \pi = p \cdot AL^{\alpha}K^{\beta} - wL - rK.$$

The profit function is just a linear transformation of the production function, therefore it's also strictly concave. The first order condition will be sufficient to get the maximum.

The FONCs are

$$\frac{\partial \pi}{\partial L} = p \cdot \alpha A L^{\alpha-1}K^{\beta} - w = 0$$
$$\frac{\partial \pi}{\partial K} = p \cdot \beta A L^{\alpha}K^{\beta-1} - r = 0$$

which gives

$$\alpha A L^{\alpha-1}K^{\beta} = \frac{w}{p} \tag{1}$$
$$\beta A L^{\alpha}K^{\beta-1} = \frac{r}{p} \tag{2}$$

Notice that there are two unknowns in these two equations, so we can solve L and K from these two equations. *There are many ways to do this, here is mine:*

Divide Equation (1) by Equation (2),

$$\frac{\alpha A L^{\alpha-1}K^{\beta}}{\beta A L^{\alpha}K^{\beta-1}} = \frac{w/p}{r/p}$$

which gives the relation between K and L as

$$K = \frac{\beta w}{\alpha r} L \tag{3}$$

Plug it into Equation (1)

$$\alpha A L^{\alpha-1}(\frac{\beta w}{\alpha r}L)^{\beta} = \frac{w}{p}$$
$$\alpha A(\frac{\beta w}{\alpha r})^{\beta} \cdot L^{\alpha+\beta-1} = \frac{w}{p}$$

which gives

$$L^*(p,w,r) = \left(\frac{w}{p\,\alpha A(\frac{\beta w}{\alpha r})^{\beta}}\right)^{\frac{1}{\alpha+\beta-1}} = \left(\frac{p\,\alpha A(\frac{\beta w}{\alpha r})^{\beta}}{w}\right)^{\frac{1}{1-\alpha-\beta}}$$

and

$$K^*(p,w,r) = \frac{\beta w}{\alpha r}L^* = \frac{\beta w}{\alpha r}\left(\frac{p\,\alpha A(\frac{\beta w}{\alpha r})^{\beta}}{w}\right)^{\frac{1}{1-\alpha-\beta}}$$
$$Y^*(p,w,r) = Y(L^*,K^*) = A(L^*)^{\alpha}(K^*)^{\beta}$$

c) Negative Cross-effect

SOSC

The second order derivative of the profit function with respect to L and K is

$$\frac{\partial^2 \pi}{\partial (L,K)^2} = p\frac{\partial^2 Y}{\partial (L,K)^2} = pH$$

where H is the Hessian matrix of the production function. Since H is negative definite and p is just a positive scalar, $\frac{\partial^2 \pi}{\partial (L,K)^2}$ *is also negative definite, therefore the second order sufficient condition hold.*

Comparative Statics

First, we show $\frac{\partial L^*}{\partial r} < 0$ as following

$$\begin{aligned}
\frac{\partial L^*}{\partial r} &= \frac{1}{1-\alpha-\beta}\left(\frac{p\,\alpha A(\frac{\beta w}{\alpha r})^\beta}{w}\right)^{\frac{1}{1-\alpha-\beta}-1} \cdot \frac{p\alpha A}{w}\beta(\frac{\beta w}{\alpha r})^{\beta-1}\cdot\frac{\beta w}{\alpha}(-\frac{1}{r^2}) \\
&= \frac{1}{1-\alpha-\beta}\left(\frac{p\,\alpha A(\frac{\beta w}{\alpha r})^\beta}{w}\right)^{\frac{1}{1-\alpha-\beta}-1} \cdot pA\beta^2(\frac{\beta w}{\alpha r})^{\beta-1}\cdot(-\frac{1}{r^2}) \\
&< 0
\end{aligned}$$

Then we want to show $\frac{\partial K^*}{\partial w} = \frac{\partial L^*}{\partial r}$. From Equation (3) we know that

$$\begin{aligned}
\frac{\partial K^*}{\partial w} &= \frac{\partial}{\partial w}(\frac{\beta w}{\alpha r}L^*) = \frac{\beta}{\alpha r}L^* + \frac{\beta w}{\alpha r}\frac{\partial L^*}{\partial w} \\
&= \frac{\beta}{\alpha r}\left(\frac{p\,\alpha A(\frac{\beta w}{\alpha r})^\beta}{w}\right)^{\frac{1}{1-\alpha-\beta}} + \frac{\beta w}{\alpha r}\frac{1}{1-\alpha-\beta}\left(\frac{p\,\alpha A(\frac{\beta w}{\alpha r})^\beta}{w}\right)^{\frac{1}{1-\alpha-\beta}-1}\cdot\frac{p\,\alpha A\cdot\beta(\frac{\beta w}{\alpha r})^{\beta-1}\cdot\frac{\beta}{\alpha r}\cdot w - p\,\alpha A(\frac{\beta w}{\alpha r})^\beta}{w^2} \\
&= \frac{1}{1-\alpha-\beta}\left(\frac{p\,\alpha A(\frac{\beta w}{\alpha r})^\beta}{w}\right)^{\frac{1}{1-\alpha-\beta}-1}\cdot\left[(1-\alpha-\beta)\frac{p\,\alpha A(\frac{\beta w}{\alpha r})^\beta}{w}\cdot\frac{\beta}{\alpha r} + \frac{\beta w}{\alpha r}\cdot\frac{p\,\alpha A\cdot\beta(\frac{\beta w}{\alpha r})^{\beta-1}\cdot\frac{\beta}{\alpha r}\cdot w - p\,\alpha A(\frac{\beta w}{\alpha r})^\beta}{w^2}\right] \\
&= \frac{1}{1-\alpha-\beta}\left(\frac{p\,\alpha A(\frac{\beta w}{\alpha r})^\beta}{w}\right)^{\frac{1}{1-\alpha-\beta}-1}\cdot\frac{pA\beta}{wr}\left[(1-\alpha-\beta)(\frac{\beta w}{\alpha r})^\beta + \beta(\frac{\beta w}{\alpha r})^{\beta-1}\cdot\frac{\beta}{\alpha r}\cdot w - (\frac{\beta w}{\alpha r})^\beta\right] \\
&= \frac{1}{1-\alpha-\beta}\left(\frac{p\,\alpha A(\frac{\beta w}{\alpha r})^\beta}{w}\right)^{\frac{1}{1-\alpha-\beta}-1}\cdot\frac{pA\beta}{wr}\left[-\alpha(\frac{\beta w}{\alpha r})^\beta\right] \\
&= \frac{1}{1-\alpha-\beta}\left(\frac{p\,\alpha A(\frac{\beta w}{\alpha r})^\beta}{w}\right)^{\frac{1}{1-\alpha-\beta}-1}\cdot\frac{pA\beta}{wr}\cdot-\alpha(\frac{\beta w}{\alpha r})^{\beta-1}\frac{\beta w}{\alpha r} \\
&= \frac{1}{1-\alpha-\beta}\left(\frac{p\,\alpha A(\frac{\beta w}{\alpha r})^\beta}{w}\right)^{\frac{1}{1-\alpha-\beta}-1}\cdot pA\beta^2(\frac{\beta w}{\alpha r})^{\beta-1}\cdot(-\frac{1}{r^2}) \\
&= \frac{\partial L^*}{\partial r} < 0
\end{aligned}$$

d) Tax

When a tax t is imposed on price p, for firms it means the price drops to $(p-t)$. In order to know its impact on the optimal output, we need to know the sign of $\frac{\partial Y^*}{\partial p}$. As

$$Y^* = A(L^*)^\alpha (K^*)^\beta = A(L^*)^\alpha \left(\frac{\beta w}{\alpha r} L^*\right)^\beta = A\left(\frac{\beta w}{\alpha r}\right)^\beta (L^*)^{\alpha+\beta},$$

So

$$\frac{\partial Y^*}{\partial p} = A\left(\frac{\beta w}{\alpha r}\right)^\beta (\alpha+\beta)(L^*)^{\alpha+\beta-1}\frac{\partial L^*}{\partial p}.$$

As we can see, the sign of $\frac{\partial Y^*}{\partial p}$ only depend on $\frac{\partial L^*}{\partial p}$.

And we know

$$\frac{\partial L^*}{\partial p} = \frac{1}{1-\alpha-\beta}\left(\frac{p\,\alpha A(\frac{\beta w}{\alpha r})^\beta}{w}\right)^{\frac{1}{1-\alpha-\beta}-1}\frac{\alpha A(\frac{\beta w}{\alpha r})^\beta}{w} > 0$$

therefore $\frac{\partial Y^*}{\partial p} > 0$. So as price decreases, the optimal output will also decrease, i.e.,

$$Y^*(p-t,w,r) = Y(L^*(p-t,w,r), K^*(p-t,w,r)) < Y(L^*(p,w,r), K^*(p,w,r)) = Y^*(p,w,r)$$

e) Short Term Reaction

See answer keys on page 3

Problem 2: Analysis on Quadratic Production Function

a) Parameter Restrictions

Given scarcity in resources including soil quality, water and fertilizer utilization, we might expect the crop production exhibits diminishing productivity. So

$$\frac{\partial^2 Y}{\partial x_1^2} < 0 \quad \Longrightarrow \frac{\partial^2 Y}{\partial x_1^2} = 2a_1 < 0$$
$$\Longrightarrow \quad a_1 < 0$$

similarly, $a_2 < 0$. The marginal product must be positive, so $b_1 > 0$, $b_2 > 0$.

b) Optimal Input under Profit Maximization

Suppose the input prices are w_1 and w_2, then the profit maximization problem is

$$\max_{x_1,x_2} \quad \pi = pY(x_1,x_2) - w_1x_1 - w_2x_2$$

The Hessian matrix of the profit function is

$$H = p\cdot\begin{bmatrix} 2a_1 & 0 \\ 0 & 2a_2 \end{bmatrix}$$

which is negative definite given the constrains derived from a) and a positive output price p. So the profit function is strictly concave and the first necessary order conditions are also sufficient.

The FONCs are

$$\frac{\partial \pi}{\partial x_1} = p(2a_1x_1 + b_1) - w_1 = 0$$
$$\frac{\partial \pi}{\partial x_2} = p(2a_2x_1 + b_2) - w_2 = 0$$

Solve them,

$$x_1^* = \frac{w_1/p - b_1}{2a_1}$$
$$x_2^* = \frac{w_2/p - b_2}{2a_2}$$

c) Optimal Input under Production Maximization

The production maximization problem is

$$\max_{x_1, x_2} \quad Y(x_1, x_2) = a_1x_1^2 + a_2x_2^2 + b_1x_1 + b_2x_2$$

The Hessian matrix of the production function is just

$$\begin{bmatrix} 2a_1 & 0 \\ 0 & 2a_2 \end{bmatrix}$$

which is negative definite given the parameter restrictions derived in part a). So the production function is strictly concave and the first order conditions are sufficient.

The FOCs are

$$\frac{\partial Y}{\partial x_1} = 2a_1x_1 + b_1 = 0$$
$$\frac{\partial Y}{\partial x_2} = 2a_2x_2 + b_2 = 0$$

Solve them,

$$x_1' = \frac{-b_1}{2a_1}; \qquad x_2' = \frac{-b_2}{2a_2}$$

Compare them with the ones under profit maximizing

$$x_1^* - x_1' = \frac{w_1/p}{2a_1} < 0; \qquad x_2^* - x_2' = \frac{w_2/p}{2a_2} < 0$$

So the optimal input level under profit maximizing is less than the one under production maximizing. The difference comes from the cost side. When the object is to maximize profit, although an increased production level will increase revenue, it will also bring an increased cost, which will drag down the profit. When the goal is only maximizing the production, the economic agent does not need to consider about the cost side.

d) The Link Between Production Function and Profit Function