

# AAE635 Problem Set #1: Suggested Answers

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## Problem 1: Chain Rule

a. Since  $y = g(h(x_1, x_2))$ ,  $h = \ln(x_1 x_2)$ , using Chain Rule

$$\frac{\partial y}{\partial x_1} = \frac{\partial g}{\partial h} \cdot \frac{\partial h}{\partial x_1} = \frac{\partial g}{\partial h} \cdot \frac{x_2}{x_1 x_2} = \frac{\partial g}{\partial h} \cdot \frac{1}{x_1} \quad (1)$$

As  $\frac{\partial g}{\partial h} \equiv \frac{\partial y}{\partial h}$ , we have

$$\frac{\partial y}{\partial h} = x_1 \cdot \frac{\partial y}{\partial x_1}$$

And by symmetry of  $x_1$  and  $x_2$ ,

$$\frac{\partial y}{\partial h} = x_2 \cdot \frac{\partial y}{\partial x_2}$$

b. Taking derivative of Equation (1) w.r.t  $x_1$  again:

$$\frac{\partial^2 y}{\partial x_1^2} = \frac{\partial^2 g}{\partial h^2} \frac{\partial h}{\partial x_1} \cdot \frac{1}{x_1} - \frac{\partial g}{\partial h} \cdot \frac{1}{x_1^2} = \frac{\partial^2 g}{\partial h^2} \cdot \frac{1}{x_1^2} - \frac{\partial g}{\partial h} \cdot \frac{1}{x_1^2}$$

As  $\frac{\partial^2 g}{\partial h^2} \equiv \frac{\partial^2 y}{\partial h^2}$ , we have

$$\frac{\partial^2 y}{\partial h^2} = x_1^2 \cdot \frac{\partial^2 y}{\partial x_1^2} + \frac{\partial g}{\partial h} = x_1^2 \cdot \frac{\partial^2 y}{\partial x_1^2} + x_1 \cdot \frac{\partial y}{\partial x_1}$$

Similarly, by symmetry of  $x_1$  and  $x_2$ ,

$$\frac{\partial^2 y}{\partial h^2} = x_2^2 \cdot \frac{\partial^2 y}{\partial x_2^2} + x_2 \cdot \frac{\partial y}{\partial x_2}$$

## Problem 2: Homogeneity

a. Multiply both  $x_1$  and  $x_2$  by  $\lambda$ , we get

$$f(\lambda x_1, \lambda x_2) = A(\lambda x_1)^{\alpha_1} (\lambda x_2)^{\alpha_2} = A \lambda^{\alpha_1 + \alpha_2} x_1^{\alpha_1} x_2^{\alpha_2} = \lambda^{\alpha_1 + \alpha_2} f(x_1, x_2)$$

So  $f(x_1, x_2)$  is homogeneous degree  $(\alpha_1 + \alpha_2)$ . And

$$\begin{aligned} x_1 f'_{x_1} + x_2 f'_{x_2} &= x_1 \cdot A \alpha_1 x_1^{\alpha_1-1} x_2^{\alpha_2} + x_2 \cdot A x_1^{\alpha_1} \alpha_2 x_2^{\alpha_2-1} \\ &= \alpha_1 A x_1^{\alpha_1} x_2^{\alpha_2} + \alpha_2 A x_1^{\alpha_1} x_2^{\alpha_2} \\ &= (\alpha_1 + \alpha_2) A x_1^{\alpha_1} x_2^{\alpha_2} \\ &= (\alpha_1 + \alpha_2) f(x_1, x_2) \end{aligned}$$

b. Similarly,

$$f(\lambda x_1, \lambda x_2) = A(\alpha_1(\lambda x_1)^\rho + \alpha_2(\lambda x_2)^\rho)^{\frac{1}{\rho}} = A(\lambda^\rho \cdot (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho))^{\frac{1}{\rho}} = \lambda A(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}} = \lambda f(x_1, x_2)$$

So  $f(x_1, x_2)$  is homogeneous degree 1. And

$$\begin{aligned} x_1 f'_{x_1} + x_2 f'_{x_2} &= x_1 \cdot A \frac{1}{\rho} (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}-1} \cdot \alpha_1 \rho x_1^{\rho-1} + x_2 \cdot A \frac{1}{\rho} (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}-1} \cdot \alpha_2 \rho x_2^{\rho-1} \\ &= A \frac{1}{\rho} (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}-1} \cdot [\alpha_1 \rho x_1^{\rho-1} + \alpha_2 \rho x_2^{\rho-1}] \\ &= A (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}-1} \cdot [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho] \\ &= A (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}} = f(x_1, x_2) \end{aligned}$$

c. I am sure you can do this one!

### Problem 3: Maximum and Minimum

a. Local Minimum and Maximum

The first order condition is

$$f'_x = 6x^2 - 6x = 6x(x - 1) = 0 \tag{2}$$

which gives

$$x = 0, \quad x = 1$$

These two points could be either local maximum or local minimum. In order to identify them, we need to use second order condition:

$$f''_{xx} = 12x - 6$$

Since

$$\begin{aligned} f''_{xx}|_{x=0} &= -6 < 0, \\ f''_{xx}|_{x=1} &= 6 > 0 \end{aligned}$$

we know  $x = 0$  is a local maximum point and  $x = 1$  is a local minimum point.

(Notice that the second order derivative does not have a constant sign over the domain, therefore we can not say that these points are global optimizers. Also notice that the second order condition is used to identify maximum or minimum points here.)

But these are only interior points, we also need to consider about the corner solutions. From Equation (2) we know  $f(x)$  is increasing ( $f'_x > 0$ ) in  $x \in [-2, 0)$ , therefore  $x = -2$  is also a local minimum. Similarly,  $f(x)$  is increasing in  $x \in (1, 2]$ , so  $x = 2$  is a local maximum.

**b. Global Minimum and Maximum**

The global maximum and minimum points are necessarily local maximum and minimum points. As we know

$$f(-2) = -28, \quad f(0) = 0, \quad f(1) = -1, \quad f(2) = 4$$

So the global maximum point is  $x = 2$  and the global minimum point is  $x = -2$ .

In fact, a plot of  $f(x)$  is

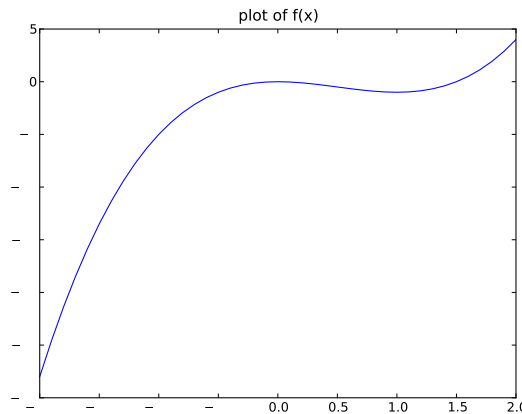


Figure 1: plot of  $f(x)$

**Problem 4: Matrix**

I will call all the matrices as  $A$  in the following notation.

- a. The determinant of this matrix is

$$\det(A) = \det \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} = 3 \times 1 - 1 \times 4 = -1$$

The inverse is

$$A^{-1} = \frac{1}{\det(A)} C^T = -1 \times \begin{bmatrix} 1 & -4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$$

**b.**

$$\det(A) = \sum_{j=1}^3 a_{3j}C_{3j} = 1 \cdot (-1)^{3+1} \det \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + 1 \cdot (-1)^{3+3} \det \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = 2 - 1 = 1$$

$$A^{-1} = \frac{1}{\det(A)} C^T = 1 \times \begin{bmatrix} -1 & 1 & -1 \\ -2 & 1 & 2 \\ 2 & -1 & -1 \end{bmatrix}^T = \begin{bmatrix} -1 & -2 & 2 \\ 1 & 1 & -1 \\ -1 & 2 & -1 \end{bmatrix}$$

**c.** This matrix is singular therefore its determinant is 0 and it's not invertible.

**d.** You should be able to solve this one now.