Suggested solutions to the Midterm Exam

1. Consider a firm using capital (K) and labor (L) to produce output y. The production possibility frontier is given by $y \le g(\mathbf{x}) = a + K + bK^2 + cKL - L^2$***, where*** $\mathbf{x} = (K, L)$***, and a, b and c are exogenous production parameters. Assume that both output and input markets are competitive at prices p and*** $\mathbf{w} = (w_K, w_L)$***, and assume interior solutions.***

a. Find the FONCs for firm's profit maximizing problem. Make necessary assumptions and justify your model set up. (12 points)

Ans: Firm's profit maximization problem is

$$\underset{K,L}{Max}\ \{py - \mathbf{wx} : y \le g(\mathbf{x}) = a + K + bK^2 + cKL - L^2; K > 0; L > 0\}$$

Assume that the prices p and $\mathbf{w} = (w_K, w_L)$ are all positive. For a given set of $\mathbf{x}$, each additional output unit, if feasible, will generate p additional profits to the firm, therefore, to maximize profits, the firm will always operate at the technology efficiency. We can rewrite the problem as

$$\underset{K,L}{Max}\ \pi = pg(\mathbf{x}) - \mathbf{wx} = p \cdot (a + K + bK^2 + cKL - L^2) - w_K K - w_L L .$$

FONCs: $\pi_{\mathbf{x}} = pg_{\mathbf{x}} - \mathbf{w} = \begin{bmatrix} \pi_K \\ \pi_L \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \pi}{\partial K} \\ \dfrac{\partial \pi}{\partial L} \end{bmatrix} = \begin{bmatrix} p \cdot (1 + 2bK + cL) - w_K \\ p \cdot (cK - 2L) - w_L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. QED.

b. Suppose that the SOSCs are satisfied. Find the comparative statics for K^* ***with respect to changes in the production parameters a, b and c. How do you expect each of the responses to be (i.e. positive, negative, or ambiguous)? (Note that you do not need to solve for the explicit function K*(p, w). Apply the implicit function theorem, make necessary assumptions if needed). (20 points)***

Ans: The SOSCs are satisfied, i.e. the Hessian matrix of the profit function, $\pi_{\mathbf{xx}} = \begin{bmatrix} \pi_{KK} & \pi_{LK} \\ \pi_{KL} & \pi_{LL} \end{bmatrix}$

$= \begin{bmatrix} 2pb & pc \\ pc & -2p \end{bmatrix}$, is negative definite. It follows that $2pb < 0$ ➔ $b < 0$, and det($\pi_{\mathbf{xx}}$)

$= -4p^2 b - p^2 c^2 > 0$ ➔ $-2\sqrt{-b} < c < 2\sqrt{-b}$.

Apply the implicit function theorem to the FONCs in part a, and denote the vector of exogenous variables a, b and c as $\boldsymbol{\beta} = (a, b, c)$, we obtain

$$\mathbf{x}_{\boldsymbol{\beta}} = \begin{bmatrix} \frac{\partial K}{\partial a} & \frac{\partial K}{\partial b} & \frac{\partial K}{\partial c} \\ \frac{\partial L}{\partial a} & \frac{\partial L}{\partial b} & \frac{\partial L}{\partial c} \end{bmatrix} = -(\pi_{\mathbf{xx}})^{-1} \pi_{\mathbf{x}\boldsymbol{\beta}} = \frac{-1}{\det(\pi_{\mathbf{xx}})} \cdot \begin{bmatrix} -2p & -pc \\ -pc & 2pb \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial \pi_K}{\partial a} & \frac{\partial \pi_K}{\partial b} & \frac{\partial \pi_K}{\partial c} \\ \frac{\partial \pi_L}{\partial a} & \frac{\partial \pi_L}{\partial b} & \frac{\partial \pi_L}{\partial c} \end{bmatrix}$$

$$= \frac{-1}{\det(\pi_{\mathbf{xx}})} \cdot \begin{bmatrix} -2p & -pc \\ -pc & 2pb \end{bmatrix} \cdot \begin{bmatrix} 0 & 2pK & pL \\ 0 & 0 & pK \end{bmatrix}.$$

Therefore, we have

$$\frac{\partial K^*}{\partial a} = \frac{-1}{\det(\pi_{\mathbf{xx}})} \cdot (-2p \cdot 0 + (-pc) \cdot 0) = 0;$$

$$\frac{\partial K^*}{\partial b} = \frac{-1}{\det(\pi_{\mathbf{xx}})} \cdot (-2p \cdot 2pK^* + (-pc) \cdot 0) = \frac{4p^2K^*}{\det(\pi_{\mathbf{xx}})} > 0; \text{ and}$$

$$\frac{\partial K^*}{\partial c} = \frac{-1}{\det(\pi_{\mathbf{xx}})} \cdot (-2p \cdot pL^* + (-pc) \cdot pK^*) = \frac{p^2(2L^* + cK^*)}{\det(\pi_{\mathbf{xx}})}$$, which can be positive or negative depending on the sign and relative size of c. Recall that $-2\sqrt{-b} < c < 2\sqrt{-b}$. If $0 < c < 2\sqrt{-b}$, then $\frac{\partial K^*}{\partial c} > 0$. If $-2\sqrt{-b} < c < 0$, then we need to check whether $c < -\frac{2L^*}{K^*}$ to determine the sign of $\frac{\partial K^*}{\partial c}$. Note that $g_{KL} = c$, i.e. c measures how the marginal product of one input is affected by the use of the other input. QED.

(***For the rest of question 1, assume SOSCs hold***.)

Now suppose that this firm is a monopsonist in the labor market (i.e. it is the only buyer of labor facing with an upward sloping supply function of labor $w_L(L)$), but is still a price taker in the capital market and the output market.

c. Find the FONCs for its profit maximizing problem. (12 points)

Ans. The firm's problem is now

$$\underset{K,L}{Max} \ \pi^m = p \cdot (a + K + bK^2 + cKL - L^2) - w_K K - w_L(L)L .$$

$$\text{FONCs: } \pi_{\mathbf{x}}^m = \begin{bmatrix} \pi_K^m \\ \pi_L^m \end{bmatrix} = \begin{bmatrix} \frac{\partial \pi^m}{\partial K} \\ \frac{\partial \pi^m}{\partial L} \end{bmatrix} = \begin{bmatrix} p \cdot (1 + 2bK + cL) - w_K \\ p \cdot (cK - 2L) - w_L - \frac{\partial w_L}{\partial L} \cdot L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \text{ QED.}$$

d. How do you expect the optimal input demand of the monopsonist, denoted by K^m and L^m, compared to those in the competitive case, i.e. K* and L* (solutions to FONC in part a)? Interpret your answer in economic intuition. (You do not need to solve for the explicit expression for K's and L's). (15 points)

Ans: Comparing the FONCs in part c to those in part a, we see that the monopsonist's FONCs has an extra term $-\frac{\partial w_L}{\partial L} \cdot L$ for π_L while π_K looks the same:

$$\pi_L^* = p \cdot g_L - w_L = 0; \qquad \text{(part a)}$$

$$\pi_L^m = p \cdot g_L - w_L - \frac{\partial w_L}{\partial L} \cdot L = 0. \qquad \text{(part c)}$$

The upward sloping supply function of labor implies $\frac{\partial w_L}{\partial L} > 0$. Thus given interior solutions of the optimal inputs, the extra term $-\frac{\partial w_L}{\partial L} \cdot L < 0$. Evaluated at K^* and L^*, $p \cdot g_L - w_L = 0$, thus $\pi_L^m < 0$. <u>If the firm increase the use of L,</u> since the production function is strictly concave (from the SOSCs), we expect the production technology exhibits diminishing MP, therefore $p \cdot g_L$ will **decrease** (note that by profit maximization with positive output and input prices, the MP is positive, $g_L >$

0); given upward sloping supply of L, w_L will **increase**; and if the new L lies in the close neighborhood of L^*, $\frac{\partial w_L}{\partial L}$ remains approximately the same, ➔ $\frac{\partial w_L}{\partial L} \cdot L$ will **increase**. Therefore, with an increase in L, π_L^m will deviate from zero furthermore (**more negative**). We can then infer that the monopsonist tends to use less L to satisfy $\pi_L^m = 0$, i.e. $L^m < L^*$.

Now we check K^m. Since $\pi_{KL} = pc > 0$ (< 0) if $c > 0$ (<0), we may infer that the monopsonist will reduce the use of K, i.e. $K^m < K^*$ if the marginal product of K increases with additional use of L ($c > 0$), and will increase the use of K, i.e. $K^m > K^*$ if the marginal product of K decreases with the additional use of L ($c < 0$). QED.

e. Suppose that the labor supply function is given as $w_L(L) = d + e\ln L$. Identify the subset of the parameter space (a, b, c, d, e, p, w) that the profit-maximizing input choices are homogenous of degree zero. Show your work and interpret. (15 points)

Ans: We observe parameters (a, b, c) in the production function only:

$$g(\mathbf{x}) = a + K + bK^2 + cKL - L^2.$$

Clearly a proportional change in any subset of (a, b, c) will not lead to proportional changes in the output level, thus these parameters are irrelevant to the homogeneity of the input demand functions.

Since $w_L(L) = d + e\ln L$, the input price of labor w_L is not exogenous while w_K is. Let $\boldsymbol{\alpha} = (p, w_K, d, e)$. Consider a proportional change in $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha}' = t\,\boldsymbol{\alpha}$, for any $t > 0$. Then, under $\boldsymbol{\alpha}'$, the profit maximization problem in part d can be written as

$$\underset{K,L}{Max}\ \pi^m = tp \cdot g(K,L) - tw_K K - (td + te\ln(L))L$$

which has for solution $K^m(t\boldsymbol{\alpha})$ and $L^m(t\boldsymbol{\alpha})$.

Given $t > 0$, note that the above equation can be written equivalently as

$$t \cdot \underset{K,L}{Max}\ \pi^m = p \cdot g(K,L) - w_K K - (d + e\ln(L))L$$

which has for solution $K^m(\boldsymbol{\alpha})$ and $L^m(\boldsymbol{\alpha})$. It follows that

$K^m(t\boldsymbol{\alpha}) = K^m(\boldsymbol{\alpha})$ and $L^m(t\boldsymbol{\alpha}) = L^m(\boldsymbol{\alpha})$ for all k > 0,

i.e. the profit maximizing input choices are homogeneous of degree zero in $\boldsymbol{\alpha} = (p, w_K, d, e)$. This means that any proportional change in prices (p, w_K), and in the intercept d and slope e of the log linear price-dependent input supply do not affect the optimal decision rules. This reflects the fact that, under profit maximization, changes that "simply rescale profit" have no impact on optimal decisions. QED. (*Note that some of you argue that the K and L should be homogeneous of degree zero on a also. However, from the FONC in part c, it should be clear that the solutions to the FONC do not contain a, i.e. the input demand functions do not have the parameter a at all. Therefore, there is no need to argue for a*)

2. Consider the following maximization problem

$$\underset{x_1,x_2}{Max}\ f(x_1,x_2) = x_1 - x_1^2 - 2x_2^2 + \theta x_1 x_2,$$

subject to $h(x_1,x_2) = \theta x_1 - 4x_2 - \theta = 0$, ***where*** θ ***is a given parameter and*** $0 < \theta < 1$.

a.** **Set up the Lagrangean equation and find the FONCs. Make necessary assumptions if needed. (12 points)

Ans: Assuming interior solutions, the Lagrangean equation is

$$L = f(x_1, x_2) + \lambda h(x_1, x_2)$$
$$= x_1 - x_1^2 - 2x_2^2 + \theta x_1 x_2 + \lambda(\theta x_1 - 4x_2 - \theta).$$

We will check: $\frac{\partial h}{\partial x_1} = \theta \neq 0$, or $\frac{\partial h}{\partial x_2} = -4 \neq 0$. Thus the CQ condition holds always.

FONCs:

$$\frac{\partial L}{\partial(x_1, x_2, \lambda)} = \begin{bmatrix} L_{x_1} \\ L_{x_2} \\ L_\lambda \end{bmatrix} = \begin{bmatrix} 1 - 2x_1 + \theta x_2 + \theta\lambda \\ -4x_2 + \theta x_1 - 4\lambda \\ \theta x_1 - 4x_2 - \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \text{QED.}$$

b.** **Show that the SONCs holds at the solutions to part a. (15 points)

Ans: Let $x_1 = x_1(x_2)$. Using the constraint, we have $x_1 = \frac{4}{\theta} x_2 + 1$. The SOC in this constrained maximization is

$$[\tfrac{\partial x_1}{\partial x_2}, 0]\ L_{\mathbf{xx}} \begin{bmatrix} \frac{\partial x_1}{\partial x_2} \\ 0 \end{bmatrix} = [\tfrac{4}{\theta} x_2 + 1, 0] \begin{bmatrix} L_{x_1 x_1} & L_{x_1 x_2} \\ L_{x_2 x_1} & L_{x_2 x_2} \end{bmatrix} \begin{bmatrix} \frac{4}{\theta} x_2 + 1 \\ 0 \end{bmatrix}.$$

Denote $\mathbf{x} = (x_1, x_2)$. Note that the Hessian matrix for the objective function is $f_{\mathbf{xx}} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} -2 & \theta \\ \theta & -4 \end{bmatrix}$. We can easily show that this is a negative definite matrix, because $f_{11} = -2 < 0$, $f_{11} = -4 < 0$, and det($f_{\mathbf{xx}}$)= $8 - \theta^2 > 0$. We can then infer that the objective function is (strictly) concave. The constraint function is linear, thus is concave (and convex also), or quasi-concave (and quasi-convex also). Using $L_{x_2} = -4x_2 + \theta x_1 - 4\lambda = 0$, and $L_\lambda = \theta x_1 - 4x_2 - \theta = 0$, we can derive that $\lambda = \frac{1}{4}\theta > 0$. Therefore, the FONCs are also sufficient, our solutions to part a will give us the global interior maxima, which implies that the SONC is always satisfied: the SOC matrix above is negative semi-definite.

(Alternatively, you can show that if the SOSC is satisfied, then the SONC will hold. It is ok to show it in the specific question as the math is simple. However, in general this approach would involve tedious math calculation, and need to solve the FONCs also. To show that the SOSC is satisfied, we can check the sign restrictions on the bordered Hessian of the Lagrangean equation, **H**. For the 2 inputs 1 constraint case, it is reduced to show det(**H**) > 0.

$$\mathbf{H} = \begin{bmatrix} L_{\mathbf{xx}} & L_{\mathbf{x}\lambda} \\ L_{\mathbf{x}\lambda} & 0 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} & L_{1\lambda} \\ L_{21} & L_{22} & L_{2\lambda} \\ L_{\lambda 1} & L_{\lambda 2} & 0 \end{bmatrix} = \begin{bmatrix} -2 & \theta & \theta \\ \theta & -4 & -4 \\ \theta & -4 & 0 \end{bmatrix},$$

➔ det(**H**) = …> 0, ➔ SOSC satisfied ➔ SONC satisfied.)