Final Exam Suggested Solutions

1. *Consider a representative consumer making consumption decision between gas (x_1) and other goods (x_2), and deriving utility according to* $U(\mathbf{x}) = 10 + \sqrt{(x_1 - 1)x_2}$.
The market price for these consumptions goods are $\mathbf{p} = (p_1, p_2) > 0$.

   a) *Using Lagrangean approach, derive the Marshallian demand functions for* x_1, x_2.

Ans: The Marshallian demand is to solve $\underset{x_1,x_2}{Max}\ U(\mathbf{x}) = 10 + \sqrt{(x_1 - 1)x_2}$, s.t. $p_1x_1 + p_2x_2 \le y$, where y is the total income budget.
Set up the Lagrangean equation: $L = 10 + \sqrt{(x_1 - 1)x_2} + \lambda(y - p_1x_1 - p_2x_2)$.
Check the CQ: Let $h(\mathbf{x})$ be the constraint function, $\partial h(\mathbf{x})/\partial x_i = -p_i \ne 0$ by assumption, thus CQ holds for this problem.
FONCs: $\partial L/\partial x_1 = ... = 0; \partial L/\partial x_2 = ... = 0; \partial L/\partial \lambda = ... = 0.$ Solve and yield:
$x_1^* = (y + p_1)/2p_1$ and $x_2^* = (y - p_1)/2p_2$.
Since the utility function is quasi-concave and prices are all positive, and if the consumer's preferences are non-satiated, the FONCs are also sufficient, and the above solutions are the Marshallian demand functions for x_1, x_2.

   b) *Find the associated expenditure function* $E(\mathbf{p}, U)$*, where U is the reference utility level (being maximized in part a) (Hint: obtain the expression for indirectly utility function, then use duality to find* $E(\mathbf{p}, U)$*).*

Ans: The indirect utility function at the utility level in part a) is:

$$V(\mathbf{p}, y) = U(\mathbf{x}^*) = 10 + \sqrt{(\frac{y + p_1}{2p_1} - 1)\cdot\frac{y - p_1}{2p_2}} = 10 + \frac{y - p_1}{2}\sqrt{\frac{1}{p_1p_2}} \Rightarrow 2(V(\mathbf{p}, y) - 10) = (y - p_1)\sqrt{\frac{1}{p_1p_2}}.$$

Given the duality $E(\mathbf{p}, V(\mathbf{p},y)) = y$, and $V(\mathbf{p}, E(\mathbf{p},U)) = U$, we have

$$2(U - 10) = (E - p_1)\sqrt{\frac{1}{p_1p_2}} \Rightarrow E(\mathbf{p}, U) = p_1 + 2(U - 10)\sqrt{p_1p_2}.$$

   c) *Obtain the Hicksian demand functions and verify the Slutsky equation.*

Ans: Use Shephard's Lemma to obtain the Hicksian demand functions: $x_i^c = \frac{\partial E}{\partial p_i}$. We have

$$x_1^c = \frac{\partial E}{\partial p_1} = 1 + (U - 10)\sqrt{\frac{p_2}{p_1}}, \text{ and } x_2^c = \frac{\partial E}{\partial p_2} = (U - 10)\sqrt{\frac{p_1}{p_2}}.$$

The Slutsky equation states that $\frac{\partial x_i^*}{\partial p_j} = \frac{\partial x_i^c}{\partial p_j} - \frac{\partial x_i^*}{\partial y}x_j^*$, for all i, j.

For i = 1, j = 1 $\frac{\partial x_1^*}{\partial p_1} = -\frac{y}{2p_1^2}$, $\frac{\partial x_1^c}{\partial p_1} = -\frac{(U-10)}{2p_1} \cdot \sqrt{\frac{p_2}{p_1}}$, and $\frac{\partial x_1^*}{\partial y} = \frac{1}{2p_1}$. We can verify the following Slutsky equation: $\frac{\partial x_1^c}{\partial p_1} - \frac{\partial x_1^*}{\partial y} x_1^* = -\frac{(U-10)}{2p_1} \cdot \sqrt{\frac{p_2}{p_1}} - \frac{1}{2p_1} \cdot \frac{(y+p_1)}{2p_1}$. Substitute in the reference utility $U = 10 + \frac{y-p_1}{2}\sqrt{\frac{1}{p_1 p_2}}$, and yield $\frac{\partial x_1^c}{\partial p_1} - \frac{\partial x_1^*}{\partial y} x_1^* = -\frac{y}{2p_1^2} = \frac{\partial x_1^*}{\partial p_1}$.

For i = 1, j = 2, $\frac{\partial x_1^*}{\partial p_2} = 0$, $\frac{\partial x_1^c}{\partial p_2} = \frac{(U-10)}{2\sqrt{p_1 p_2}}$. We can verify the following Slutsky equation: $\frac{\partial x_1^c}{\partial p_2} - \frac{\partial x_1^*}{\partial y} x_2^* = \frac{(U-10)}{2\sqrt{p_1 p_2}} - \frac{1}{2p_1} \cdot \frac{(y-p_1)}{2p_2}$. Substitute in the reference utility $U = 10 + \frac{y-p_1}{2}\sqrt{\frac{1}{p_1 p_2}}$, and yield $\frac{\partial x_1^c}{\partial p_2} - \frac{\partial x_1^*}{\partial y} x_2^* = \frac{y-p_1}{4p_1 p_2} - \frac{y}{4p_1 p_2} = 0 = \frac{\partial x_1^*}{\partial p_2}$.

Similarly we can verify the Slutsky equation for i = 2, j = 1 and i = 2, j = 2.

d) *Suppose an oil crisis occurs, the government needs to consider some short term policy to restrict the gas consumption. There are two options: price control or quantity control. The government also considers offering a lump sum income tax credit to consumers so that they will vote yes to the policy intervention. Now you are hired by the government to provide advice regarding which policy intervention the government should choose and how much should be offered to consumers as income tax credit. Illustrate how you will conduct this consulting assignment. State your decision criteria for your recommendations.*

Ans: First we need to know what government's target level of gas consumption is. With that information, we can infer the corresponding level of price changes (given we have full information about the market). Since the gas consumption will stay at the status quo level if consumers do not vote "yes", this implies that the appropriate welfare measurement should be the compensating variation (CV). We shall calculate the amount of CV under the price control policy and the CV under quantity control. Our recommendation of policy will be based on cost-effectiveness, i.e, we will recommend the government to choose the policy that involves less income tax credit (equals to the lower CV). Suppose income stays the same in the short run. To calculate CV for price control, we use the formula $CV = E(\mathbf{p}_0, U_0) - E(\mathbf{p}_1, U_0) = \int_{\mathbf{p}_1}^{\mathbf{p}_0} x^c(p, U_0)dp$, where subscript 1 stands for after change situation, and "0" stands for the status quo. To calculate CV for quantity control, we use the formula $CV = B(\mathbf{x}_1, U_0) - B(\mathbf{x}_0, U_0) = \int_{\mathbf{x}_0}^{\mathbf{x}_1} p(\mathbf{x}, U_0)d\mathbf{x}$.

2. *You are evaluating a firm that chooses two inputs (x_1, x_2) to minimize cost*

$$C(y, \mathbf{w}) = \underset{\mathbf{x}}{Min}\ \{\mathbf{wx} : y = g(\mathbf{x}), \mathbf{x} \in R^2, \mathbf{x} \geq 0\},$$

where $\mathbf{w} = (w_1, w_2)$ *is the price vector for* $\mathbf{x}$, $y = g(\mathbf{x})$ *is the production function.*

a. *Suppose that the firm adopts a new technology that improves the productivity of* x_2 *such that each unit of* x_2 *now is equivalent to* tx_2 *before, t > 1, i.e. the production function is now*

$y = g(x_1, tx_2)$. *Using Lagrangean approach, illustrate how to obtain the cost minimizing input demand functions* $x_i^c(y, \mathbf{w}, t)$ *, i = 1, 2. Make necessary assumptions when needed.*

Ans: The firm faces with problem: $C(y,\mathbf{w}) = \underset{\mathbf{x}}{Min}\ \{\mathbf{wx} : y = g(x_1, tx_2), \mathbf{x} \in R^2, \mathbf{x} \geq 0\}$. CQ requires that $\frac{\partial g(x_1, tx_2)}{\partial x_i} \neq 0$ for at least one input, and we assume this condition holds.

The Lagrangean equation is $L = \mathbf{wx} + \lambda(y - g(x_1, tx_2))$, assuming that the non-negativity conditions on **x** are not binding. The FONCs are: $\partial L/\partial x_1 = w_1 - \lambda \cdot \partial g/\partial x_1 = 0;\ \partial L/\partial x_2 = w_2 - t\lambda \cdot \partial g/\partial x_2 = 0;$ and $\partial L/\partial \lambda = y - g(x_1, tx_2) = 0.$ Solve this system of equation will yield candidate solutions for optimal input demand uses. For simplicity we assume that the production function $g(\mathbf{x})$ is quasi-concave, and input prices are positive, this will generate the special case where the FONCs are also sufficient for the global interior minimum solution.

b. *Use the envelop theorem to evaluate* $\frac{\partial C}{\partial t}$*, interpret the results.*

Ans: According to the envelop theorem, $\frac{\partial C}{\partial t} = \frac{\partial L}{\partial t} = \frac{\partial L}{\partial g} \cdot \frac{\partial g}{\partial x_2} \cdot x_2^c$. From the Lagrangean equation, we have $\frac{\partial L}{\partial g} = -1$; From the FONCs, we have $\frac{\partial g}{\partial x_2} = \frac{w_2}{t\lambda^c} > 0$ given assumptions that input prices are positive, CQ condition holds, and $t > 1$; Finally $x_2^c > 0$ given assumption that the non-negativity conditions on **x** are not binding (see part a). Therefore, $\frac{\partial C}{\partial t} = \frac{\partial L}{\partial g} \cdot \frac{\partial g}{\partial x_2} \cdot x_2^c < 0$, suggesting that when technology improvement generate more efficient production (t goes up), the total cost of producing given amount of output will go down.

c. *You find that the output elasticities* $\frac{\partial \ln(x_i^c)}{\partial \ln(y)}$ *are the same for both inputs. You also find that the input price elasticities* $\frac{\partial \ln(x_i^c)}{\partial \ln(\mathbf{w})}$ *are independent of output y. What does this imply for the production technology? Justify your answer and discuss the implications.*

Ans: This may imply that the production technology is homothetic. With homothetic technology, the production function satisfies $y = g(\mathbf{x}) = F(h(\mathbf{x}))$, where $F(h)$ is a strictly increasing function, and $h(\mathbf{x})$ is a function that is homogeneous of degree one in **x**. This implies that the indirect cost function can be written as $C(y, \mathbf{w}) = G(y) \cdot C(1, \mathbf{w})$. From Shephards' lemma, this implies that $\frac{\partial C}{\partial w_i} = x_i^c = G(y) \cdot \frac{\partial C(1,\mathbf{w})}{\partial w_i}$, and $\ln(x_i^c) = \ln(G(y)) + \ln(\frac{\partial C(1,\mathbf{w})}{\partial w_i})$, i = 1, 2. As a result, the output elasticities of cost minimizing input demand functions are $\frac{\partial \ln(x_1^c)}{\partial \ln(y)} = \frac{\partial \ln G(y)}{\partial \ln y} = \frac{\partial \ln(x_2^c)}{\partial \ln(y)}$, the same for all inputs, and price elasticities are $\frac{\partial \ln x_i^c}{\partial \ln \mathbf{w}} = \frac{\partial \ln \frac{\partial C(1,\mathbf{w})}{\partial w_i}}{\partial \ln \mathbf{w}}$, independent of output y. If the production is homothetic, it also implies that the marginal rate of substitution between any two inputs is homogeneous of degree zero in **x**, and the output elasticities are equal to the inverse of the scale elasticity SE.

d. *Suppose you find that the indirect cost function has the form*

$$C(y, \mathbf{w}, t) = 3w_1^2 + w_2 + (4y^2 + 5)(2w_1 + \tfrac{10w_2}{t^2}).$$

The firm currently produces 100 units of output (y = 100) facing prices w_1 = 2 and w_2 = 1 and using technology t = 10. Find the scale elasticity for current production. Under free entry and exit, do you have any recommendation to this firm?

Ans: The scale elasticity SE can be evaluated by AC/MC. The average cost AC = $C(y, \mathbf{w}, t)/y$ = (3x4+1+(4x100x100+5)(2x2+10/100))/100=1640, and the marginal cost MC = $\partial C(y, \mathbf{w}, t)/\partial y$ = $8y(2w_1 + \tfrac{10w_2}{t^2})$ =8x100x(2x2+10/100)=3280. ➔ SE = AC/MC = 0.50. So the firm operates at a decreasing return to scale region. Under free entry and exit, the market price will be driven down to MC = AC, the firm is producing too much at the current level (where AC < MC), we would suggest the firm to lower its production level to the point where MC = AC. However, if other entrants are not as efficient in production as this firm is, i.e. others' minimal AC (= MC) is higher than this firm, then it is still possible for this firm to operate in the AC < MC region.

3. *In Adam Smith's world, the "invisible hands" work, thus free market ideology is promoted. This is consistent with the first and second welfare theorems.*

   a. *What are the first and second welfare theorems (you can use your own words)? Under what conditions will these theorems hold and why?*

Ans: Check your notes to find the first and second welfare theorems (or use your own words). Specify those assumptions we made along the way to get to the competitive equilibrium slide, make it clear which assumptions are need for what purpose (e.g., those for establishing a P.E. allocation, those for establishing P.E. = zero maximal allocation, those for establishing saddle point solutions, and those needed for establishing competitive equilibrium = PE).

   b. *Do you expect invisible hands work in today's economy? Explain and illustrate examples where they likely work vs. not.*

Ans: They work if all the conditions in part a hold, and not if some conditions are violated. You should be able to come up with specific examples of where they work vs. not, and be specific on how they related to the conditions you specified in part a.

Have a nice winter break!