AAE 635
Suggested solutions to final exam - Fall 2008

1) *You are investigating a firm that chooses two inputs to minimize cost*

$$C(y,\mathbf{w},\beta) = \underset{\mathbf{x}}{Min}\ \{\mathbf{wx} : y = (1+\beta)g(\mathbf{x}), \mathbf{x} \in R^2, \mathbf{x} \geq 0\},$$

where $\mathbf{w} = (w_1, w_2)$ *is the price vector for* $\mathbf{x}$, $y = (1+\beta)g(\mathbf{x})$ *is the production function, where* $\beta > 0$ *is a technology enhancement index.*

a) Using Lagrangean approach, illustrate how to obtain the cost minimizing input demand functions $x_i^c(y,\mathbf{w},\beta)$, *i = 1, 2. Make necessary assumptions when needed and explain your assumptions. (10 points)*

Ans: Set up the Lagrangean equation as $L(\mathbf{x},\lambda) = \mathbf{wx} + \lambda(y-(1+\beta)g(\mathbf{x}))$. First check that the CQ condition holds such that there exists at least one x_i such that $\frac{\partial(y-(1+\beta)g(\mathbf{x}))}{\partial x_i} \neq 0$, ➔ $MP_i \neq 0$.
Since the marginal product has to be non-negative to be consistent with cost minimization, we can conclude that when CQ holds, there exist at least one input with positive marginal product, $MP_i > 0$.
We then derive the FONCs:

$$L_{\mathbf{x}}(\mathbf{x},\lambda) = \mathbf{w} - \lambda(1+\beta)g_{\mathbf{x}}(\mathbf{x}) = 0;$$
$$L_{\lambda}(\mathbf{x},\lambda) = y-(1+\beta)g(\mathbf{x}) = 0.$$

Solve them to yield $\mathbf{x}^c(y,\mathbf{w},\beta)$ and $\lambda^c(y,\mathbf{w},\beta)$.
FONCs are also sufficient if the objective function is convex, the constraint function is quasi-convex, and $\lambda^c(y,\mathbf{w},\beta) > 0$. The objective function = $\mathbf{wx}$, which is linear thus both concave and convex; the constraint function = $y-(1+\beta)g(\mathbf{x})$, which is quasi-convex if the production function $g(\mathbf{x})$ is quasi-concave, implying diminishing MRS; and $L_{\mathbf{x}} = 0 \Rightarrow \lambda = \frac{w_i}{(1+\beta)MP_i} > 0$ if assuming positive input prices (we already know $\beta > 0$ and $MP_i > 0$ by CQ). Therefore, under positive input prices and quasi-concave production technology, FONCs are also sufficient, our solutions to the FONCs, $\mathbf{x}^c(y,\mathbf{w},\beta)$ are the cost minimizing input demand functions.

b) The firm faces technological progress represented by an increase in β. *Evaluate and discuss the properties of* $\mathbf{x}_{\mathbf{w}}^c \equiv \frac{\partial \mathbf{x}^c}{\partial \mathbf{w}}$, $\mathbf{x}_y^c \equiv \frac{\partial \mathbf{x}^c}{\partial y}$, *and* $\mathbf{x}_\beta^c \equiv \frac{\partial \mathbf{x}^c}{\partial \beta}$. *Interpret your results (Hint: find primal-dual results, discuss symmetric restrictions and sign restrictions...) (20 points)*

Ans: Let $\boldsymbol{\alpha} = (\mathbf{w},\beta,y)$, and $h(\mathbf{x},\boldsymbol{\alpha}) = y-(1+\beta)g(\mathbf{x})$. The primal-dual results give:

$[L_{\boldsymbol{\alpha}\mathbf{x}}, L_{\boldsymbol{\alpha}\lambda}]\begin{bmatrix}\mathbf{x}_{\boldsymbol{\alpha}}^c \\ \lambda_{\boldsymbol{\alpha}}^c\end{bmatrix}$ is symmetric and negative semi-definite matrix subject to constraint with $h_{\boldsymbol{\alpha}}\mathbf{u} = 0$,

where $\mathbf{u}$ is a 4x1 vector and we have $\mathbf{u}^T[L_{\boldsymbol{\alpha}\mathbf{x}}, L_{\boldsymbol{\alpha}\lambda}]\begin{bmatrix}\mathbf{x}_{\boldsymbol{\alpha}}^c \\ \lambda_{\boldsymbol{\alpha}}^c\end{bmatrix}\mathbf{u} = \mathbf{u}^T\begin{bmatrix}L_{\mathbf{wx}} & L_{\mathbf{w}\lambda} \\ L_{\beta\mathbf{x}} & L_{\beta\lambda} \\ L_{y\mathbf{x}} & L_{y\lambda}\end{bmatrix}\begin{bmatrix}\mathbf{x}_{\mathbf{w}}^c & \mathbf{x}_\beta^c & \mathbf{x}_y^c \\ \lambda_{\mathbf{w}}^c & \lambda_\beta^c & \lambda_y^c\end{bmatrix}\mathbf{u} \leq 0$.

Since $h_{\boldsymbol{\alpha}} = (0, 0, -g(\mathbf{x}), 1)$, we can choose $\mathbf{u}^T = (u_1, u_2, u_3, g(\mathbf{x})u_3)$, where u_1, u_2, u_3 can take any value, and we will have $h_{\boldsymbol{\alpha}}\mathbf{u} = 0$. We can rewrite $\mathbf{u} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & g(\mathbf{x}) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and plug it back to the primal dual results above:

$$\mathbf{u}^T [L_{\boldsymbol{\alpha}\mathbf{x}}, L_{\boldsymbol{\alpha}\lambda}] \begin{bmatrix} \mathbf{x}^c_{\boldsymbol{\alpha}} \\ \lambda^c_{\boldsymbol{\alpha}} \end{bmatrix} \mathbf{u} = [u_1, u_2, u_3] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & g(\mathbf{x}) \end{bmatrix} \begin{bmatrix} L_{\mathbf{wx}} & L_{\mathbf{w}\lambda} \\ L_{\beta\mathbf{x}} & L_{\beta\lambda} \\ L_{y\mathbf{x}} & L_{y\lambda} \end{bmatrix} \begin{bmatrix} \mathbf{x}^c_{\mathbf{w}} & \mathbf{x}^c_{\beta} & \mathbf{x}^c_{y} \\ \lambda^c_{\mathbf{w}} & \lambda^c_{\beta} & \lambda^c_{y} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & g(\mathbf{x}) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

$$= [u_1, u_2, u_3] \begin{bmatrix} L_{\mathbf{wx}} & L_{\mathbf{w}\lambda} \\ L_{\beta\mathbf{x}} + g(\mathbf{x}) L_{y\mathbf{x}} & L_{\beta\lambda} + g(\mathbf{x}) L_{y\lambda} \end{bmatrix} \begin{bmatrix} \mathbf{x}^c_{\mathbf{w}} & \mathbf{x}^c_{\beta} + g(\mathbf{x})\mathbf{x}^c_{y} \\ \lambda^c_{\mathbf{w}} & \lambda^c_{\beta} + g(\mathbf{x})\lambda^c_{y} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \le 0.$$

Since $L_{\mathbf{wx}} = \mathbf{I}_2$, $L_{\mathbf{w}\lambda} = L_{y\mathbf{x}} = 0$, $L_{\beta\mathbf{x}} = -\lambda g_{\mathbf{x}}$, $L_{\beta\lambda} = -g$, and $L_{y\lambda} = 1$, the above results can be rewritten as

$$\mathbf{u}^T [L_{\boldsymbol{\alpha}\mathbf{x}}, L_{\boldsymbol{\alpha}\lambda}] \begin{bmatrix} \mathbf{x}^c_{\boldsymbol{\alpha}} \\ \lambda^c_{\boldsymbol{\alpha}} \end{bmatrix} \mathbf{u} = [u_1, u_2, u_3] \begin{bmatrix} \mathbf{I}_2 & 0 \\ -\lambda g_{\mathbf{x}} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}^c_{\mathbf{w}} & \mathbf{x}^c_{\beta} + g(\mathbf{x})\mathbf{x}^c_{y} \\ \lambda^c_{\mathbf{w}} & \lambda^c_{\beta} + g(\mathbf{x})\lambda^c_{y} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= [u_1, u_2, u_3] \begin{bmatrix} \mathbf{x}^c_{\mathbf{w}} & \mathbf{x}^c_{\beta} + g(\mathbf{x})\mathbf{x}^c_{y} \\ -\lambda g_{\mathbf{x}} \mathbf{x}^c_{\mathbf{w}} & -\lambda g_{\mathbf{x}} (\mathbf{x}^c_{\beta} + g(\mathbf{x})\mathbf{x}^c_{y}) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \le 0.$$

Since u_1, u_2, u_3 can take any value, it follows that $\begin{bmatrix} \mathbf{x}^c_{\mathbf{w}} & \mathbf{x}^c_{\beta} + g(\mathbf{x})\mathbf{x}^c_{y} \\ -\lambda g_{\mathbf{x}} \mathbf{x}^c_{\mathbf{w}} & -\lambda g_{\mathbf{x}} (\mathbf{x}^c_{\beta} + g(\mathbf{x})\mathbf{x}^c_{y}) \end{bmatrix}$ is symmetric and negative semi-definite (without any constraints).

It follows that

1) $\mathbf{x}^c_{\mathbf{w}} \equiv \frac{\partial \mathbf{x}^c}{\partial \mathbf{w}}$ is symmetric and negative semi-definite, ➔ $\frac{\partial x^c_i}{\partial w_i} \le 0$: in general downward sloping input demand curve; $\frac{\partial x^c_i}{\partial w_j} = \frac{\partial x^c_j}{\partial w_i}$: cross price responses are symmetric between two inputs;

2) $-\lambda g_{\mathbf{x}} (\mathbf{x}^c_{\beta} + g(\mathbf{x})\mathbf{x}^c_{y}) \le 0$ ➔ $g_{\mathbf{x}} (\mathbf{x}^c_{\beta} + g(\mathbf{x})\mathbf{x}^c_{y}) \ge 0$; and

3) $-\lambda g_{\mathbf{x}} \mathbf{x}^c_{\mathbf{w}} = \mathbf{x}^c_{\beta} + g(\mathbf{x})\mathbf{x}^c_{y}$.

We can not sign $\mathbf{x}^c_{y}$ and $\mathbf{x}^c_{\beta}$ directly but can establish the relationship between the two via 2) and 3) above.

c) *Show that the Allen elasticity of substitution between input 1 and input 2 is necessarily positive (has a lower bound of zero). (Hint: identify homogeneity of* $x^c_i(y, \mathbf{w}, \beta)$*, then use the Euler theorem…). What is the economic interpretation? (10 points)*

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Ans: First we can infer that $\mathbf{x}^c(y,\mathbf{w},\beta)$ is homogeneous of degree 0 in **w**, as

$$\mathbf{x}^c(y,\mathbf{w},\beta) = \arg\min_{\mathbf{x}} \{\mathbf{wx} : y = (1+\beta)g(\mathbf{x}), \mathbf{x}\in R^2, \mathbf{x}\geq 0\}$$

$$= \arg\min_{\mathbf{x}} \{k\mathbf{wx} : y = (1+\beta)g(\mathbf{x}), \mathbf{x}\in R^2, \mathbf{x}\geq 0\}, \text{ where } k \text{ is any constant}$$

$$= \mathbf{x}^c(y,k\mathbf{w},\beta).$$

By Euler theorem, $\mathbf{x}^c_{\mathbf{w}}\mathbf{w} = 0$, ➔ $\frac{\partial x_i^c}{\partial w_i} w_i + \frac{\partial x_i^c}{\partial w_j} w_j = 0$ ➔ $\frac{\partial x_i^c}{\partial w_j} = -\frac{\partial x_i^c}{\partial w_i}\cdot\frac{w_i}{w_j} \geq 0$ because $\frac{\partial x_i^c}{\partial w_i} \leq 0$ from part b) and input prices **w** are positive from part a). The AES between input 1 and 2 is $\sigma_{12} = \frac{\partial x_1^c}{\partial w_2}\cdot\frac{TC}{x_1^c x_2^c} \geq 0$, i.e. it has a lower bound of zero. This implies that when there are only two inputs in the production technology, they cannot be complements to each other. In another word, they are in general viewed as substitutes.

2) *Consider a consumer making consumption decisions between gas (x_1) and other goods (x_2). He/she derives utility according to* $U(\mathbf{x}) = 4\sqrt{x_1 x_2}$ *. The market price for these consumption goods are* $\mathbf{p} = (p_1, p_2) > 0$. *Consider the case where the consumption decisions are made in order to achieve a certain utility level* U_0.

a) Derive the Hicksian demand functions for both gas and other goods. (10 points)

Ans: Consumer's problem is $\underset{\mathbf{x}}{Min}\ \mathbf{px}$ s.t. $U_0 = 4\sqrt{x_1x_2}$. Since CQ is clearly satisfied, we can set up the Lagrangean equation $L(\mathbf{x},\lambda) = \mathbf{px} + \lambda(U_0 - 4\sqrt{x_1x_2})$. Solving the FONCs

$$\begin{cases} L_{x_1}(\mathbf{x},\lambda) = p_1 - 4\lambda\frac{\sqrt{x_2}}{2\sqrt{x_1}} = 0 \\ L_{x_2}(\mathbf{x},\lambda) = p_2 - 4\lambda\frac{\sqrt{x_1}}{2\sqrt{x_2}} = 0 \\ L_\lambda(\mathbf{x},\lambda) = U_0 - 4\sqrt{x_1x_2} = 0 \end{cases} \text{ yields } \begin{cases} x_1^c = \frac{U^0}{4}\sqrt{\frac{p_2}{p_1}} \\ x_2^c = \frac{U^0}{4}\sqrt{\frac{p_1}{p_2}} \end{cases}.$$

Since the objective function is linear thus convex, the constraint function is quasi-convex, and $\lambda = \frac{p_i}{MU_i} > 0$, the FONCs are also sufficient. Thus the solutions above are the Hicksian demand functions for both gas and other goods.

*b) Find the associated indirect utility function V(**p**, y), where y is income (<u>Hint</u>: obtain the expression for the expenditure function, then use duality to find V(**p**, y)). (10 points)*

Ans: Plug the Hicksian demand function derived in part a) into the expenditure, we obtain the expenditure function $E(\mathbf{p},U^0) = p_1x_1^c + p_2x_2^c = p_1\cdot\frac{U^0}{4}\cdot\sqrt{\frac{p_2}{p_1}} + p_2\cdot\frac{U^0}{4}\cdot\sqrt{\frac{p_1}{p_2}} = \frac{U^0}{2}\sqrt{p_1p_2}$. From the duality $E(\mathbf{p},V(\mathbf{p},y)) = y$, we can write $\frac{V(\mathbf{p},y)}{2}\sqrt{p_1p_2} = y$ ➔ $V(\mathbf{p},y) = \frac{2y}{\sqrt{p_1p_2}}$.

c) Use Roy's identity to obtain the Marshallian demand functions and verify the Slutsky equation. (10 points)

Ans: Roy's identity: $x_1^*(\mathbf{p},y) = -\frac{\partial v/\partial p_1}{\partial v/\partial y} = -\frac{\frac{2y}{\sqrt{p_2}}(-\frac{1}{2})p_1^{-1.5}}{2/\sqrt{p_1p_2}} = \frac{y}{2p_1}$; similarly $x_2^*(\mathbf{p},y) = \frac{y}{2p_2}$.

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The Slutsky equation is: $\frac{\partial x_i^*}{\partial p_j} = \frac{\partial x_i^c}{\partial p_j} - \frac{\partial x_i^*}{\partial y} x_j^*$. For x_1, we will verify that $\frac{\partial x_1^*}{\partial p_1} = \frac{\partial x_1^c}{\partial p_1} - \frac{\partial x_1^*}{\partial y} x_1^*$ and $\frac{\partial x_1^*}{\partial p_2} = \frac{\partial x_1^c}{\partial p_2} - \frac{\partial x_1^*}{\partial y} x_2^*$.

We have $\frac{\partial x_1^*}{\partial p_1} = -\frac{y}{2p_1^2}$, and $\frac{\partial x_1^c}{\partial p_1} - \frac{\partial x_1^*}{\partial y} x_1^* = -\frac{U^0}{8p_1}\sqrt{\frac{p_2}{p_1}} - \frac{1}{2p_1} \cdot \frac{y}{2p_1}$. Since $E(\mathbf{p}, V(\mathbf{p}, y)) = y = \frac{U^0}{2}\sqrt{p_1 p_2}$, we have $-\frac{U^0}{8p_1}\sqrt{\frac{p_2}{p_1}} - \frac{1}{2p_1} \cdot \frac{y}{2p_1} = -\frac{y}{4p_1^2} - \frac{y}{4p_1^2} = -\frac{y}{2p_1^2}$, thus $\frac{\partial x_1^*}{\partial p_1} = \frac{\partial x_1^c}{\partial p_1} - \frac{\partial x_1^*}{\partial y} x_1^*$.

We have $\frac{\partial x_1^*}{\partial p_2} = 0$, and $\frac{\partial x_1^c}{\partial p_2} - \frac{\partial x_1^*}{\partial y} x_2^* = \frac{U^0}{8}\sqrt{\frac{1}{p_1 p_2}} - \frac{1}{2p_1} \cdot \frac{y}{2p_2}$. Again by $E(\mathbf{p}, V(\mathbf{p}, y)) = y = \frac{U^0}{2}\sqrt{p_1 p_2}$, we have $\frac{U^0}{8}\sqrt{\frac{1}{p_1 p_2}} - \frac{1}{2p_1} \cdot \frac{y}{2p_2} = \frac{y}{4p_1 p_2} - \frac{y}{4p_1 p_2} = 0$, thus $\frac{\partial x_1^*}{\partial p_2} = \frac{\partial x_1^c}{\partial p_2} - \frac{\partial x_1^*}{\partial y} x_2^*$.

Similarly we can verify the Slutsky equation for x_2.

d) *Given the decreasing price of gas, the government launched a program aiming to reduce the carbon dioxide emissions, i.e., imposing a carbon tax of $0.56 per gallon gas. If the consumer would like to participate in a group action lobbying the government to give up this policy (it does not mean that consumers do not care about green house gas, they may think there are better ways to deal with it), how much would he/she like to contribute to the lobbying fund? Assume the gas price before the tax policy is $1.69 per gallon, and the price of other goods are normalized to 1 and the household income is normalized to $72. (10 points)*

Ans: Since the policy will be in effect if the consumers do not take action, suggesting that the property rights lies in the “change”, thus we will use the EV measurement. The tax is $0.56, price before tax is $1.69 and after tax is $2.25. Given total income $72 and price of other goods at $1, we can find the utility level after tax $U_1 = \frac{2y}{\sqrt{p_1 p_2}} = \frac{144}{\sqrt{2.25}} = \frac{144}{1.5} = 96$.

$EV = E(p_0, U_1) - E(p_1, U_1) = \frac{U_1}{2}\sqrt{1.69} - \frac{U_1}{2}\sqrt{2.25} = 62.4 - 72 = -9.6.$ Thus the consumer is willing to give $9.6 to the lobby fund.

e) *Now instead of tax policy, the government imposes a quantity control policy, i.e. each consumer is rationed to two thirds of his/her historical consumption (at price $1.69 per gallon), how would your answer change compared to part d)? Illustrate how you derive answers here. (10 points)*

Ans: The only change between d) and e) is that quantity control replaces price intervention. Thus we should still use EV measurement. Before policy the consumer buy $x_1^0(\mathbf{p}, y) = \frac{y}{2p_1} = \frac{72}{2 \cdot 1.69} = 21.3$, after policy the consumer can only consume x_1^1 =21.3x2/3=14.2 gallons.

$EV = B(x_1^1, U_1) - B(x_1^0, U_1) = \int_{x_1^0}^{x_1^1} \frac{\partial B(x, U_1)}{\partial x_1} dx_1 = \int_{21.3}^{14.2} p_1(\mathbf{x}, U_1) dx_1$. Note that the inverse demand $p_1(\mathbf{x}, U_1)$ can be obtained from $\underset{\mathbf{p}}{Min} \ \{\mathbf{p}^T\mathbf{x} - E(\mathbf{p}, U) : \mathbf{p}^T\mathbf{g} = 1, \mathbf{p} \geq 0\}$. In this case, the consumer is willing to give $EV = \int_{21.3}^{14.2} p_1(\mathbf{x}, U_1) dx_1$ to the lobby fund.

f) *Given the information available so far (from both questions), would you expect the “invisible hand” to work or not in this economy? Discuss and justify your answer. (Hint: define necessary terms first, and then comment…). (10 points)*

Ans: See you notes.

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